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James Richards

A COURSE OF MATHEMATICS,

48602

IN TWO VOLUMES:

COMPOSED, AND MORE ESPECIALLY DESIGNED,

FOR THE USE OF THE GENTLEMEN CADETS IN THE ROYAL MILITARY ACADEMY AT WOOLWICH.

BY

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Of

PLANE

PLANE TRIGONOMETRY.

DEFINITIONS.

1. PLANE TRIGONOMETRY treats of the relations and calculations of the ſides and angles of plane triangles.

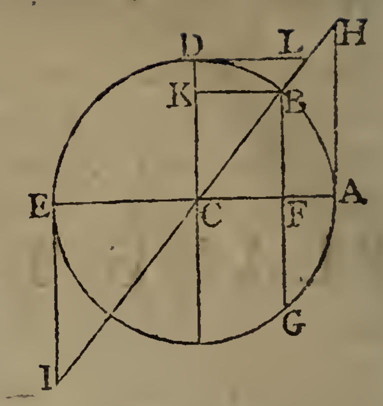
2. The circumference of every circle (as before obſerved in Geom. Def. 56) is ſuppoſed to be divided into 360 equal parts, called Degrees; alſo each degree into 60 Minutes, each minute into 60 Seconds, and ſo on.

Hence a ſemicircle contains 180 degrees, and a quadrant 90 degrees.

3. The Meaſure of any angle (Def. 57, Geom.) is an arc of any circle contained between the two lines which form that angle, the angular point being the centre; and it is eſtimated by the number of degrees contained in that arc.

Hence, a right angle, being meaſured by a quadrant, or quarter of the circle, is an angle of 90 degrees; and the ſum of the three angles of every triangle, or two right angles, is equal to 180 degrees. Therefore, in a right-angled triangle, taking one of the acute angles from 90 degrees, leaves the other acute angle; and the ſum of two angles, in any triangle, taken from 180 degrees, leaves the third angle; or one angle being taken from 180 degrees, leaves the ſum of the other two angles.

4. Degrees are marked at the top of the figure with a ſmall °, minutes with ′, ſeconds with ″, and ſo on. Thus, 57° 30′ 12″, denote 57 degrees 30 minutes and 12 ſeconds.



5. The Complement of an arc, is what it wants of a quadrant or 90°. Thus, if AD be a quadrant, then BD is the complement of the arc AB; and, reciprocally, AB is the complement of BD. So that, if AB be an arc of 50°, then its complement BD will be 40°.

6. The Supplement of an arc, is what it wants of a ſemicircle, or 180°. Thus, if ADE be a ſemicircle, then BDE is the ſupplement of the arc AB; and reciprocally, AB is the ſupplement of the arc BDE. So that, if AB be an arc of 50°, then its ſupplement BDE will be 130°.

7. The Sine, or Right Sine, of an arc, is the line drawn from one extremity of the arc, perpendicular to the diameter which paſſes through the other extremity. Thus, BF is the ſine of the arc AB, or of the arc BDE.

Corol. Hence the ſine (BF) is half the chord (BG) of the double arc (BAG).

8. The Verſed Sine of an arc, is the part of the diameter intercepted between the arc and its ſine. So, AF is the verſed ſine of the arc AB, and EF the verſed ſine of the arc EDB.

9. The Tangent of an arc, is a line touching the circle in one extremity of that arc, continued from thence to meet a line drawn from the centre through the other extremity: which laſt line is called the Secant of the ſame arc. Thus, AH is the tangent, and CH the ſecant, of the arc AB. Alſo, EI is the tangent, and CI the ſecant, of the ſupplemental arc BDE. And this latter tangent and ſecant are equal to the former, but are accounted negative, as being drawn in an oppoſite or contrary direction to the former.

10. The Coſine, Cotangent, and Coſecant, of an arc, are the ſine, tangent, and ſecant of the complement of that arc, the Co being only a contraction of the word complement. Thus, the arcs AB, BD being the complements of each other, the ſine, tangent or ſecant of the one of theſe, is the coſine, cotangent or coſecant of the other. So, BF, the ſine of AB, is the coſine of BD; and BK, the ſine of BD, is the coſine of AB: in like manner, AH, the tangent of AB, is the cotangent of BD; and DL, the tangent of DB, is the cotangent of AB: alſo, CH, the ſecant of AB, is the coſecant of BD; and CL, the ſecant of BD, is the coſecant of AB.

Corol. Hence ſeveral remarkable properties eaſily follow from theſe definitions; as,

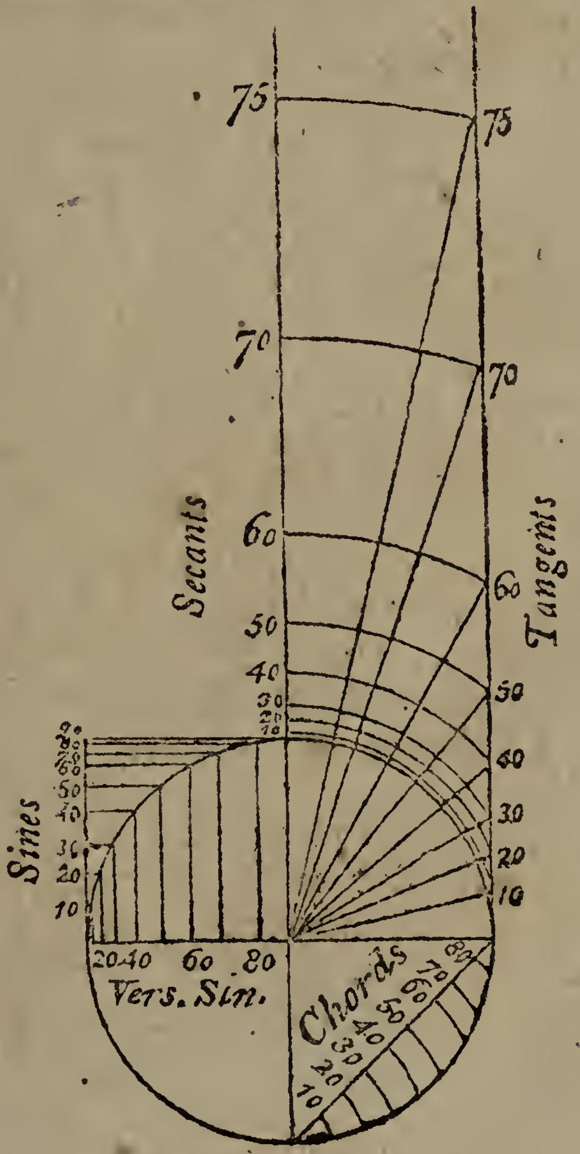
1*ſt*, That an arc and its ſupplement have the ſame ſine, tangent, and ſecant; but the two latter, the tangent and ſecant, are accounted negative when the arc is greater than a quadrant or 90 degrees.

2*d*, When the arc is 0, or nothing, the ſine and tangent are nothing, but the ſecant is then the radius CA.—But when the arc is a quadrant AD, then the ſine is the greateſt it can be, being the radius CD of the circle; and both the tangent and ſecant are infinite.

3*d*, Of any arc AB, the verſed ſine AF, and coſine BK, or CF, together make up the radius CA of the circle.—The radius CA, tangent AH, and ſecant CH, form a right angled triangle CAH. So alſo do the radius, ſine, and coſine, form another right-angled triangle CBF or CBK. As alſo the radius, cotangent and coſecant, another right-angled triangle CDL. And all theſe right-angled triangles are ſimilar to each other.

11. The ſine, tangent, or ſecant of an angle, is the ſine, tangent, or ſecant of the arc by which the angle is meaſured, or of the degrees, &c. in the ſame arc or angle.

12. The method of conſtructing the ſcales of chords, ſines, tangents, and ſecants, uſually engraven on inſtruments, for practice, is exhibited in the annexed figure.



13. A Trigonometrical Canon, is a table exhibiting the length of the ſine, tangent, and ſecant, to every degree and minute of the quadrant, with reſpect to the radius, which is expreſſed by unity or 1, and conceived to be divided into 10000000 or more decimal parts. And farther, the logarithms of theſe ſines, tangents, and ſecants are alſo ranged in the tables; which are moſt commonly uſed, as they perform the calculations by only addition and ſubtraction, inſtead of the multiplication

and diviſion by the natural ſines, &c, according to the nature of logarithms.

Upon this table depends the numeral ſolution of the ſeveral caſes in trigonometry. It will therefore be proper to begin with the mode of conſtructing it, which may be done in the following manner:

PROBLEM I.

To find the Sine and Coſine of a Given Arc.

THIS problem is reſolved after various ways. One of theſe is as follows, viz. by means of the ratio between the diameter and circumference of a circle, together with the known ſeries for the ſine and coſine, hereafter demonſtrated. Thus, the ſemicircumference of the circle, whoſe radius is 1, being 3·14159265358979 3 &c, the proportion will therefore be,

as the number of degrees or minutes in the ſemicircle,
is to the degrees or minutes in the propoſed arc,
ſo is 3·14159265 &c, to the length of the ſaid arc.

This length of the arc being denoted by the letter a; and its ſine and coſine by s and c; then will theſe two be expreſſed by the two following ſeries, viz.

$$s = a - \frac{a^3}{2.3} + \frac{a^5}{2.3.4.5} - \frac{a^7}{2.3.4.5.6.7} + \&c.$$
$$= a - \frac{a^3}{6} + \frac{a^5}{120} - \frac{a^7}{5040} + \&c.$$
$$c = 1 - \frac{a^2}{2} + \frac{a^4}{2.3.4} - \frac{a^6}{2.3.4.5.6} + \&c.$$
$$= 1 - \frac{a^2}{2} + \frac{a^4}{24} - \frac{a^6}{720} + \&c.$$

Example 1. If it be required to find the ſine and coſine of one minute. Then, the number of minutes in 180° being 10800, it will be firſt, as 10800 : 1 :: 3·14159265 &c. : ·000290888208665 = the length of an arc of one minute. Therefore, in this caſe,

a = ·0002908882
and $\frac{1}{6}a^3$ = ·0000000000004 &c,
the diff. is s = ·0002908882 the ſine of 1 minute.
Alſo, from 1.
take $\frac{1}{2}a^2$ = 0·0000000423079 &c,
leaves c = ·9999999577 the coſine of 1 minute.

Exam-

Example 2. For the ſine and coſine of 5 degrees. Here, as $180° : 5° :: 3.14159265$ &c. $: .08726646 = a$ the length of 5 degrees. Hence $a = .08726646$

$$- \tfrac{1}{6}a^3 = - .00011076$$
$$+ \tfrac{1}{120}a^5 = .00000004$$

theſe collected give $s = .08715574$ the ſine of 5°.

And, for the coſine, $1 = 1.$

$$- \tfrac{1}{2}a^2 = - .00380771$$
$$+ \tfrac{1}{24}a^4 = .00000241$$

theſe collected give $c = .99619470$ the coſine of 5°.

After the ſame manner, the ſine and coſine of any other arc may be computed. But the greater the arc is, the ſlower the ſeries will converge, in which caſe a greater number of terms muſt be taken to bring out the concluſion to the ſame degree of exactneſs.

Or, having found the ſine, the coſine will be found from it, by the property of the right-angled triangle CBF, viz. the coſine $CF = \sqrt{CB^2 - BF^2}$, or $c = \sqrt{1 - s^2}$.

There are alſo other methods of conſtructing the canon of ſines and coſines, which, for brevity's ſake, are here omitted.

PROBLEM II.

To compute the Tangents and Secants.

THE ſines and coſines being known, or found by the foregoing problem; the tangents and ſecants will be eaſily found, from the principle of ſimilar triangles, in the following manner:

In the firſt figure, where, of the arc AB, BF is the ſine, CF or BK the coſine, AH the tangent, CH the ſecant, DL the cotangent, and CL the coſecant, the radius being CA or CB or CD; the three ſimilar triangles CFB, CAH, CDL give the following proportions:

1*ſt*, CF : FB :: CA : AH; whence the tangent is known, being a fourth proportional to the coſine, ſine, and radius.

2*d*, CF : CB :: CA : CH; whence the ſecant is known, being a third proportional to the coſine and radius.

3d, BF : FC :: CD : DL; whence the cotangent is known, being a fourth proportional to the fine, cofine, and radius.

4th, BF : BC :: CD : CL; whence the cofecant is known, being a third proportional to the fine and radius.

Having given an idea of the calculation of fines, tangents and fecants, we may now proceed to refolve the feveral cafes of Trigonometry; previous to which, however, it may be proper to add a few preparatory notes and obfervations, as below.

Note 1. There are ufually three methods of refolving triangles, or the cafes of trigonometry; namely, Geometrical Conftruction, Arithmetical Computation, and Inftrumental Operation.

In the Firft Method, The triangle is conftructed, by making the parts of the given magnitudes, namely the fides from a fcale of equal parts, and the angles from a fcale of chords, or by fome other inftrument. Then, meafuring the unknown parts, by the fame fcales or inftruments, the folution will be obtained near the truth.

In the Second Method, Having ftated the terms of the proportion according to the proper rule or theorem, refolve it like any other proportion, in which a fourth term is to be found from three given terms, by multiplying the fecond and third together, and dividing the product by the firft, in working with the natural numbers: or, in working with the logarithms, add the logs. of the fecond and third terms together, and from the fum take the log. of the firft term; then the natural number anfwering to the remainder is the fourth term fought.

In the Third Method, Or Inftrumentally, as fuppofe by the log. lines on one fide of the common two-foot fcales; Extend the Compaffes from the firft term, to the fecond or third, which happens to be of the fame kind with it; then that extent will reach from the other term to the fourth term, as required, taking both extents towards the fame end of the fcale.

Note 2. Every triangle has fix parts, viz. three fides and three angles. And in every triangle, or cafe in trigonometry, there muft be given three parts, to find the other three. Alfo, of the three parts that are given, one of them at leaft muft be a fide; becaufe, with the fame angles the fides may be greater or lefs in any proportion.

Note 3.

Note 3. All the cafes in trigonometry, may be comprifed in three varieties only; viz.

1*ft*, When a fide and its oppofite angle are given.
2*d*, When two fides and the contained angle are given.
3*d*, When the three fides are given.

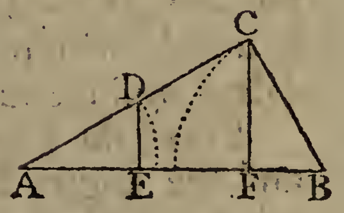
For there cannot poffibly be more than thefe three varieties of cafes; for each of which it will therefore be proper to give a feparate theorem, as follows:

THEOREM I.

When a Side and its Oppofite Angle, are two of the Given Parts.

THEN the fides of the triangle have the fame proportion to each other, as the fines of their oppofite angles have.

That is, As any one fide,
Is to the fine of its oppofite angle;
So is any other fide,
To the fine of its oppofite angle.



Demonftr. For, let ABC be the propofed triangle, having AB the greateft fide, and BC the leaft. Take AD = BC, confidering it as a radius; and let fall the perpendiculars DE, CF, which will evidently be the fines of the angles A and B, to the radius AD or BC. But the triangles ADE, ACF are equiangular, and therefore have their like fides proportional, namely, AC : CF :: AD or BC : DE; that is, AC is to the fine of its oppofite angle B, as BC to the fine of its oppofite angle A.

Note 1. In practice, to find an angle, begin the proportion with a fide oppofite a given angle. And to find a fide, begin with an angle oppofite a given fide.

Note 2. An angle found by this rule is ambiguous, or uncertain whether it be acute or obtufe, unlefs it be a right angle, or unlefs its magnitude be fuch as to prevent the ambiguity; becaufe the fine anfwers to two angles, which are fupplements to each other: and accordingly the geometrical conftruction forms two triangles with the fame parts that are given, as in the example below; and when there is no reftriction or limitation included in the queftion, either of them may be taken. The degrees in the table, anfwering to the fine, is the acute angle; but if the angle be obtufe, fubtract

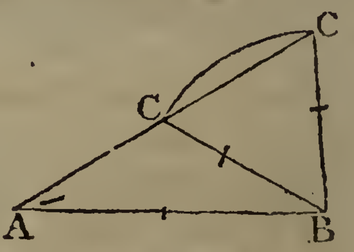
ſubtract thoſe degrees from 180°, and the remainder will be the obtuſe angle. When a given angle is obtuſe, or a right one, there can be no ambiguity; for then neither of the other angles can be obtuſe, and the geometrical conſtruction will form only one triangle.

EXAMPLE I.

In the plane triangle ABC,

Given $\begin{cases} AB\ 345 \text{ yards} \\ BC\ 232 \text{ yards} \\ \angle A\ 37^\circ\ 20' \end{cases}$

Required the other parts.



1. *Geometrically.*

Draw an indefinite line, upon which ſet off AB = 345, from ſome convenient ſcale of equal parts.—Make the angle A = $37^\circ\frac{1}{3}$.—With a radius of 232, taken from the ſame ſcale of equal parts, and centre B, croſs AC in the two points C, C.—Laſtly, join BC, BC, and the figure is conſtructed, which gives two triangles, and ſhewing that the caſe is ambiguous.

Then, the ſides AC meaſured by the ſcale of equal parts, and the angles B and C meaſured by the line of chords, or other inſtrument, will be found to be nearly as below; viz.

AC	174	∠ B	27°	∠ C	$115^\circ\frac{1}{2}$
or	$374\frac{1}{2}$	or	$78\frac{1}{4}$	or	$64\frac{1}{2}$

2. *Arithmetically.*

Firſt, to find the angles at C.

As ſide	BC	232				log.	2·3654880
To ſin. op.	∠A	37°	20′				9·7827958
So ſide	AB	345					2·5378191
To ſin. op.	∠C	115°	36′ or	64°	24′		9·9551269
add	∠A	37	20	37	20		
the ſum		152	56 or	101	44		
taken from		180	00	180	00		
leaves	∠ B	27	04 or	78	16		

Then,

Then, to find the side AC.

As sine	∠A 37° 20′	- -	log. 9·7827958
To op. side BC	232	- -	2·3654880
So sin. ∠ B	{ 27° 04′	- -	9·6580371
	{ 78 16	- -	9·9908291
To op. side AC	174·07	- -	2·2407293
or	374·56	- -	2·5735213

3. *Instrumentally.*

In the first proportion.—Extend the compasses from 232 to 345 upon the line of numbers; then that extent will reach, on the sines, from 37°⅓ to 64°½, the angle C.

In the second proportion.—Extend the compasses from 37°⅓ to 27° or 78°¼, on the sines; then that extent will reach, on the line of numbers, from 232 to 174 or 374½, the two values of the side AC.

EXAMPLE II.

In the plane triangle ABC,

Given	AB 365 poles ∠ A 57° 12′ ∠ B 24 45	Ans.	∠ C 98° 3′ AC 154·33 BC 309·86

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given	AC 120 feet BC 112 feet ∠ A 57° 27′	Ans.	∠ B 64° 34′ 21″ or 115 25 39 ∠ C 57 58 39 or 7 7 21 AB 112·65 feet or 16·47 feet.

Required the other parts.

THEOREM II.

When two Sides and their Contained Angle are given.

Then it will be,

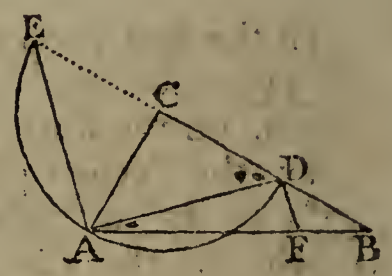
As the sum of those two sides,
Is to the difference of the same sides;
So is the tang. of half the sum of their op. angles,
To the tang. of half the diff. of the same angles.

Hence,

Hence, becaufe it has been fhewn (Alg. p. 243), that the half fum of any two quantities increafed by their half difference, gives the greater, and diminifhed by it gives the lefs, if the half difference of the angles, fo found, be added to their half fum, it will give the greater angle, and fubtracting it will leave the lefs angle.

Then all the angles being now known, the unknown fide will be found by the former theorem.

Demonft. Let ABC be the propofed triangle, having the two given fides AC, BC, including the given angle C. With the centre C, and radius CA, the lefs of thefe two fides, defcribe a femicircle, meeting the other fide BC produced in D and E. Join AE, AD, and draw DF parallel to AE.



Then, BE is the fum, and BD the diff. of the two given fides CB, CA. Alfo, the fum of the two angles CAB, CBA, is equal to the fum of the two CAD, CDA, thefe fums being each the fupplement of the vertical angle C to two right angles: but the two latter CAD, CDA are equal to each other, being oppofite to the two equal fides CA, CD: hence either of them, as CDA, is equal to half the fum of the two unknown angles CAB, CBA. Again, the exterior angle CDA is equal to the two interior angles B and DAB; therefore the angle DAB is equal to the diff. between CDA and B, or between CAD and B: confequently the fame angle DAB is equal to half the difference of the unknown angles B and CAB; of which it has been fhewn that CDA is the half fum.

Now the angle DAE, in a femicircle, is a right angle, or AE is perpendicular to AD; and DF, parallel to AE, is alfo perpendicular to AD: confequently AE is the tangent of CDA the half fum, and DF the tangent of DAB the half difference of the angles, to the fame radius AD, by the definition of a tangent. But the tangents AE, DF being parallel, it will be as BE : BD :: AE : DF; that is, as the fum of the fides is to the difference of the fides, fo is the tangent of half the fum of the oppofite angles, to the tangent of half their difference.

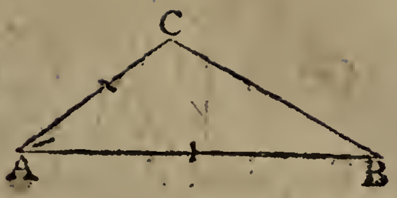
Note. The fum of the unknown angles is found, by taking the given angle from 180°.

EXAM-

EXAMPLE I.

In the plane triangle ABC,

Given { AB 345 yards; AC 174·07 yards; ∠A 37° 20′



Required the other parts.

1. *Geometrically.*

Draw AB = 345 from a ſcale of equal parts. Make the angle A = 37° 20′. Set off AC = 174 by the ſcale of equal parts. Join BC, and it is done.

Then the other parts being meaſured, they are found to be nearly as follow; viz. the ſide BC 232 yards, the angle B 27°, and the angle C 115°½.

2. *Arithmetically.*

As ſum of ſides AB, AC, - -	519·07	log.	2·7152259
To diff. of ſides AB, AC, - -	170·93	-	2·2328183
So tang. half ſum ∠s C and B -	71° 20′	-	10·4712979
To tang. half diff. ∠s C and B -	44 16	-	9·9888903
their ſum gives ∠ C	115 36		
their diff. gives ∠ B	27 4		

Then, by the former theorem,

As ſin. ∠ C 115° 36′ or 64° 24′ -	log.	9·9551259
To its op. ſide AB 345 - - -		2·5378191
So ſin. ∠ A 37° 20′ - - -		9·7827958
To its op. ſide BC 232 - - -		2·3654890

3. *Inſtrumentally.*

In the firſt proportion.—Extend the compaſſes from 519 to 171, on the line of numbers; then that extent will reach, on the tangents, from 71°⅓ (the contrary way, becauſe the tangents are ſet back again from 45°) a little beyond 45, which being ſet ſo far back from 45, falls upon 44°¼, the fourth term.

In the ſecond proportion.—Extend from 64°⅓ to 37°⅓, on the ſines; then that extent will reach, on the numbers, from 345 to 232, the fourth term ſought.

EXAM-

EXAMPLE II.

In the plane triangle ABC,

Given	AB 365 poles AC 154·33 ∠ A 57°. 12′	Anſ.	BC 309·86 ∠ B 24° 45 ∠ C 98 3

Required the other parts.

EXAMPLE III.

In the plane triangle ABC,

Given	AC 120 yards BC 112 yards ∠ C 57° 58′ 39″	Anſ.	AB 112·65 ∠ A 57° 27′ 0″ ∠ B 64 34 21

Required the other parts.

THEOREM III.

When the Three Sides of the Triangle are given.

THEN, having let fall a perpendicular from the greateſt angle upon the oppoſite ſide, or baſe, dividing it into two ſegments, and the whole triangle into two right-angled triangles; it will be,

As the baſe, or ſum of the ſegments,
Is to the ſum of the other two ſides;
So is the difference of thoſe ſides,
To the diff. of the ſegments of the baſe.

Then half the difference of the ſegments being added to the half ſum, or the half baſe, gives the greater ſegment; and the ſame ſubtracted gives the leſs ſegment.

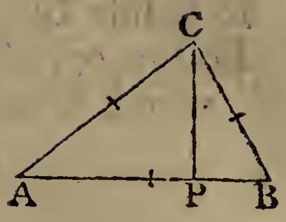
Hence, in each of the two right-angled triangles, there will be known two ſides, and the angle oppoſite to one of them; conſequently the other angles will be found by the firſt theorem.

Demonſtr. By cor. to theor. 35, Geom. the rectangle under the ſum and difference of the two ſides, is equal to the rectangle under the ſum and difference of the two ſegments. Therefore, by forming the ſides of theſe rectangles into a proportion, it will appear that the ſums and differences are proportional as in this theorem, by theor. 76, Geometry.

EXAM-

EXAMPLE I.

In the plane triangle ABC,



Given the fides $\begin{cases} AB\ 345 \text{ yards} \\ AC\ 232 \\ BC\ 174{\cdot}07 \end{cases}$

To find the angles.

1. *Geometrically.*

Draw the bafe AB = 345 by a fcale of equal parts. With radius 232, and centre A, defcribe an arc; and with radius 174, and centre B, defcribe another arc, cutting the former in C. Join AC, BC, and it is done.

Then, by meafuring the angles, they will be found to be nearly as follows, viz.

$\angle$ A 27°, $\angle$ B 37°$\frac{1}{3}$, and $\angle$ C 115°$\frac{1}{2}$.

2. *Arithmetically.*

Having let fall the perpendicular CP, it will be,
As the bafe AB : AC + BC :: AC — BC : AP — BP,
that is, as 345 : 406·07 :: 57·93 : 68·18 = AP — BP,
its half is - 34·09
the half bafe is - 172·50

the fum of thefe is 206·59 = AP
and their diff. 138·41 = BP

Then, in the triangle APC, right-angled at P,

As the fide AC	- -	232	- log.	2·3654880
To fin. op. $\angle$ P	- -	90°	- -	10·0000000
So is fide AP	- -	206·59	- -	2·3151093
To fin. op. $\angle$ ACP	-	62° 56′	-	9·9496213
which taken from	-	90 00	-	

leaves the $\angle$ A 27 04

Again,

Again, in the triangle BPC, right-angled at P,

As the ſide BC	- - 174·07	- log.	2·2407239
To ſin. op. ∠ P	- - 90°	- -	10·0000000
So is ſide BP	- - 138·41	- -	2·1411675
To ſin. op. ∠ BCP	- 52° 40′	- -	9·9004436
which taken from	- 90 00		
leaves the ∠ B	37 20		

Alſo, the	- ∠ ACP	62°	56′
added to	- ∠ BCP	52	40
gives the whole	∠ ACB	115	36

So that all the three angles are as follow, viz.

the ∠ A 27° 4′; the ∠ B 37° 20′; the ∠ C 115° 36′.

3. *Inſtrumentally.*

In the firſt proportion.—Extend the compaſſes from 345 to 406, on the line of numbers; then that extent will reach, on the ſame line, from 58 to 68·2 nearly, which is the difference of the ſegments of the baſe.

In the ſecond proportion.—Extend from 232 to 206½, on the line of numbers; then that extent will reach, on the ſines, from 90° to 63°.

In the third proportion.—Extend from 174 to 138½; then that extent will reach from 90° to 52⅔° on the ſines.

EXAMPLE II.

In the plane triangle ABC,

Given the ſides	AB 365 poles	Anſ.	∠ A	57° 12′
	AC 154·33		∠ B	24 45
	BC 309·86		∠ C	98 3

To find the angles.

In the plane triangle ABC,

Given the ſides	AB 120	Anſ.	∠ A	57° 27′ 00″
	AC 112·6		∠ B	57 58 39
	BC 112		∠ C	64 34 21

To find the angles.

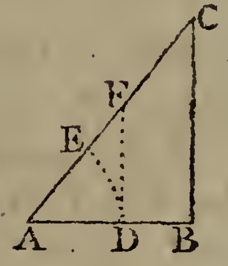
The

The three foregoing theorems include all the cafes of plane triangles, both right-angled and oblique; befides which, there are other theorems fuited to fome particular forms of triangles, which are fometimes more expeditious in their ufe than the general ones; one of which, as the cafe for which it ferves fo frequently occurs, may be here taken, as follows:

THEOREM IV.

When, in a Right-angled Triangle, there are given one Leg and the Angles; to find the other Leg or the Hypothenufe; it will be,

As radius, (i. e. fin. of 90° or tang. of 45°)
Is to the given leg,
So is tang. of its adjacent angle
To the other leg;
And fo is fecant of the fame angle
To the hypothenufe.



Demonftr. AB being the given leg, in the right-angled triangle ABC; with the centre A, and any affumed radius AD, defcribe an arc DE, and draw DF perpendicular to AB, or parallel to BC. Now it is evident, from the definitions, that DF is the tangent, and AF the fecant, of the arc DE, or of the angle A which is meafured by that arc, to the radius AD. Then, becaufe of the parallels BC, DF, it will be, - - as AD : AB :: DF : BC :: AF : AC, which is the fame as the theorem is in words.

EXAMPLE I.

In the right-angled triangle ABC,

Given $\left\{\begin{array}{l}\text{the leg AB } 162 \\ \angle \text{ A } 53^\circ\ 7'\ 48''\end{array}\right\}$ To find AC and BC.

1. *Geometrically.*

Make AB = 162 equal parts, and the angle A = 53° 7′ 48″; then raife the perpendicular BC, meeting AC in C. So fhall AC meafure 270, and BC 216.

2. *Arith-*

2. *Arithmetically.*

As radius	tang. 45°	log.	10·0000000
To leg AB	162		2·2095150
So tang. ∠ A	53° 7′ 48″		10·1249371
To leg BC	216		2·3344521
So secant ∠ A	53° 7′ 48″		10·2218477
To hyp. AC	270		2·4313627

3. *Instrumentally.*

Extend the compasses from 45° to 53°$\frac{1}{8}$, on the tangents. Then that extent will reach from 162 to 216 on the line of numbers.

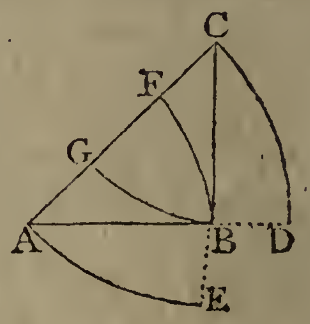
EXAMPLE II.

In the right-angled triangle ABC,

Given $\left\{\begin{array}{l}\text{the leg AB 180}\\ \text{the} \angle \text{ A } 62° 40'\end{array}\right.$ Ans. $\left\{\begin{array}{ll}\text{AC} & 392{\cdot}0146\\ \text{BC} & 348{\cdot}2464\end{array}\right.$

To find the other two sides.

Note. There is sometimes given another method for right-angled triangles, which is this:



ABC being such a triangle, make one leg AB radius, that is, with centre A, and distance AB, describe an arc BF. Then it is evident that the other leg BC represents the tangent, and the hypothenuse AC the secant, of the arc BF, or of the angle A.

In like manner, if the leg BC be made radius; then the other leg AB will represent the tangent, and the hypothenuse AC the secant, of the arc BG or angle C.

But if the hypothenuse be made radius; then each leg will represent the sine of its opposite angle; namely, the leg AB the sine of the arc AE or angle C, and the leg BC the sine of the arc CD or angle A.

And then the general rule for all these cases, is this, namely, that the sides of the triangle bear to each other the same proportion as the parts which they represent.

And this is called, Making every side radius.

OF

Of HEIGHTS and DISTANCES, &c.

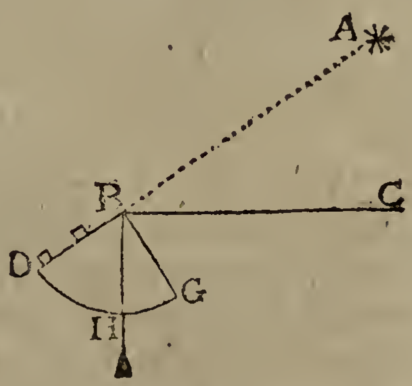
BY the menſuration and protraction of lines and angles, are determined the lengths, heights, depths, and diſtances of bodies or objects.

Acceſſible lines are meaſured by applying to them ſome certain meaſure a number of times, as an inch, or foot, or yard. But inacceſſible lines muſt be meaſured by taking angles, or by ſuch-like method, drawn from the principles of geometry.

When inſtruments are uſed for taking the magnitude of the angles in degrees, the lines are then calculated by trigonometry: in the other methods, the lines are calculated from the principle of ſimilar triangles, without regard to the meaſure of the angles.

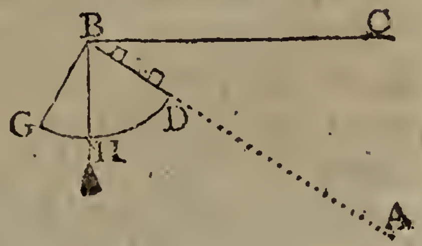
Angles of elevation, or of depreſſion, are uſually taken either with a theodolite, or with a quadrant, divided into degrees and minutes, and furniſhed with a plummet ſuſpended from the centre, and two ſights fixed on one of the radii, or elſe with teleſcopic ſights.

To take an Angle of Altitude and Depreſſion with the Quadrant.



Let A be any object, as the ſun, moon, or a ſtar, or the top of a tower, or hill, or other eminence: and let it be required to find the meaſure of the angle ABC, which a line drawn from the object makes above the horizontal line BC.

Fix the centre of the quadrant in the angular point, and move it round there as a centre till, with one eye at D, the other being ſhut, you perceive the object A through the ſights: then will the arc GH of the quadrant, cut off by the plumb line BH, be the meaſure of the angle ABC as required.



The angle ABC of depreſſion of any object A, below the horizontal line BC, is taken in the ſame manner; except that here the eye is applied to the centre, and the meaſure of the angle is the arc GH, on the other ſide of the plumb line.

The following examples are to be conſtructed and calculated by the foregoing methods, treated of in Trigonometry.

EXAMPLE I.

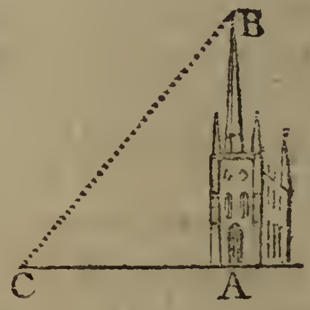
Having meaſured a diſtance of 200 feet, in a direct horizontal line, from the bottom of a ſteeple, the angle of elevation of its top, taken at that diſtance, was found to be 47° 30′: from hence it is required to find the height of the ſteeple.

Conſtruction.

Draw an indefinite line, upon which ſet off AC = 200 equal parts, for the meaſured diſtance. Erect the indefinite perpendicular AB; and draw CB ſo as to make the angle C = 47° 30′, the angle of elevation; and it is done. Then AB, meaſured on the ſcale of equal parts, is nearly $218\frac{1}{4}$.

Calculation.

As radius	- -	10·0000000
To AC 200	- -	2·3010300
So tang. ∠C 47° 30′		10·0379475
To AB 218·26 required		2·3389775



EXAMPLE II.

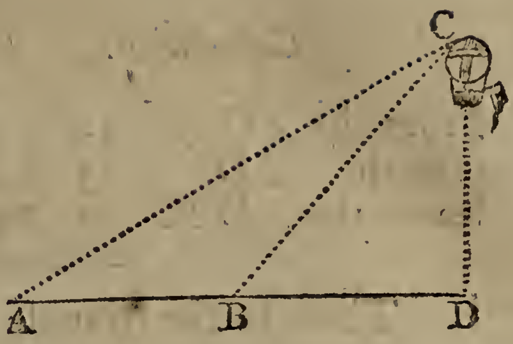
What was the perpendicular height of a cloud, or of a balloon, when its angles of elevation were 35° and 64°, as taken by two obſervers, at the ſame time, both on the ſame ſide of it, and in the ſame vertical plane; the diſtance between them being half a mile or 880 yards. And what was its diſtance from the ſaid two obſervers?

Conſtruction.

Draw an indefinite ground line, upon which ſet off the given diſtance AB = 880; then A and B are the places of the obſervers. Make the angle A = 35°, and the angle B = 64°; and the interſection of the lines at C will be the place of the balloon: from whence the perpendicular CD, being let fall, will be its perpendicular height. Then, by meaſurement, are found the diſtances and height nearly as follows, viz. AC 1631, BC 1041, DC 936.

Cal-

Calculation.



Firſt, from	∠ B	64°
take	∠ A	35
leaves	∠ ACB	29

Then, in the triangle ABC,

As ſin. ∠ACB	29°	9·6855712
To op. ſide AB	880	2·9444827
So ſin. ∠ A	35°	9·7585913
To op. ſide BC	1041·125	3·0175028

As ſin. ∠ACB	29°	9·6855712
To op. ſide AB	880	2·9444827
So ſin. ∠ B	116° or 64°	9·9536602
To op. ſide AC	1631·442	3·2125717

And, in the triangle BCD,

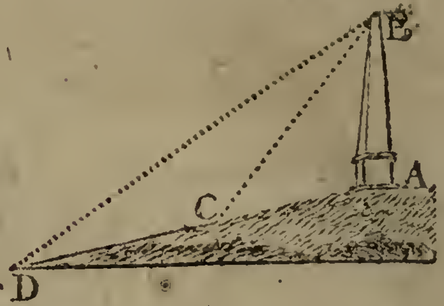
As ſin. ∠ D	90°	10·0000000
To op. ſide BC	1041·125	3·0175028
So ſin. ∠ B	64°	9·9536602
To op. ſide CD	935·757	2·9711630

EXAMPLE III.

Having to find the height of an obeliſk ſtanding on the top of a declivity, I firſt meaſured from its bottom a diſtance of 40 feet, and there found the angle, formed by the oblique plane and a line imagined to go to the top of the obeliſk, 41°; but, after meaſuring on in the ſame direction 60 feet further, the like angle was only 23° 45′. What then was the height of the obeliſk?

Conſtruction.

Draw an indefinite line for the ſloping plane or declivity, in which aſſume any point A for the bottom of the obeliſk, from whence ſet off the diſtance AC = 40, and again CD = 60 equal parts. Then make the angle C = 41°, and the angle D = 23° 45′; and the point B where the two lines meet will be the top of the obeliſk. Therefore AB, joined, will be its height.



Calculation.

From the ∠C	41°	00′
take the ∠D	23	45
leaves the ∠DBC	17	15

Then, in the triangle DBC,

As ſin. ∠DBC	17° 15′	- - -	9·4720856
To op. ſide DC	60	- - -	1·7781513
So ſin. ∠D	23 45	- - -	9·6050320
To op. ſide CB	81·488	- - -	1·9110977

And, in the triangle ABC,

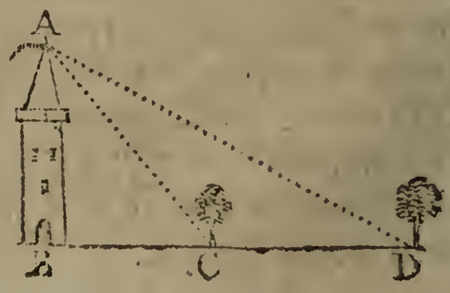
As ſum of ſides CB, CA	121·488	-	2.0845334
To diff. of ſides CB, CA	41·488	-	1·6179225
So tang. half ſum ∠s A, B	69° 30′	-	10·4272623
To tang. half diff. ∠s A, B	42 24½	-	9·9606514
the diff. of theſe is ∠CBA	27 5½		

Laſtly, as ſin. ∠CBA	27° 5′½	- -	9·6582842
To op. ſide CA	40	- -	1·6020600
So ſin. ∠C	41° 0′	- -	9·8169429
To op. ſide AB	57·623	- -	1·7607187

EXAMPLE IV.

Wanting to know the diſtance between two inacceſſible trees, or other objects, from the top of a tower, 120 feet high, which lay in the ſame right line with the two objects, I took the angles formed by the perpendicular wall and lines conceived to be drawn from the top of the tower to the bottom of each tree, and found them to be 33° and 64°½. What then may be the diſtance between the two objects?

Conſtruction.



Draw the indefinite ground line BD, and perpendicular to it BA = 120 equal parts. Then draw the two lines AC, AD making the two angles BAC, BAD equal to the given

given angles 33° and 64°½. So ſhall C and D be the places of the two objects.

Calculation.

Firſt, In the right-angled triangle ABC,

As radius	-	- - - - -	10·0000000
To AB	-	120 - - -	2·0791812
So tang. ∠BAC		33° - - -	9·8125174
To BC	-	77·929 - - -	1·8916986

And, in the right-angled triangle ABD,

As radius	-	- - - - -	10·0000000
To AB	- -	120 - -	2·0791812
So tang. ∠BAD	-	64°½ - -	10·3215039
To BD	- -	251·585 - -	2·4006851
from which take BC		77·929	
leaves the diſt. CD		173·656 as required.	

EXAMPLE V.

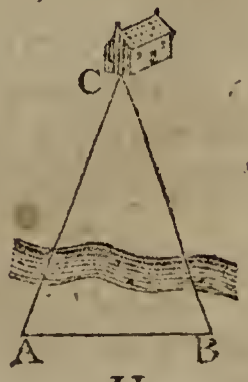
Being on the ſide of a river, and wanting to know the diſtance to a houſe which was ſeen on the other ſide, I meaſured 200 yards in a ſtraight line by the ſide of the river; and then at each end of this line of diſtance took the horizontal angle formed between the houſe and the other end of the line; which angles were, the one of them 68° 2′, and the other 73° 15′. What then were the diſtances from each end to the houſe?

Conſtruction.

Draw the line AB = 200 equal parts. Then draw AC ſo as to make the angle A = 68° 2′, and BC to make the angle B = 73° 15′. So ſhall the point C be the place of the houſe required.

Calculation.

To the given ∠A	68°	2′
add the given ∠B	73	15
then their ſum	141	17
being taken from	180	0
leaves the third ∠C	38	43



Hence

Hence,	As ſin. ∠C	38° 43′	-	9·7962062
	To op. ſide AB	200	-	2·3010300
	So ſin. ∠A	68° 2′	-	9·9672679
	To op. ſide BC	296·54	-	2·4720917

And,	As ſin. ∠C	38° 43′	-	9·7962062
	To op. ſide AB	200	-	2·3010300
	So ſin. ∠B	73° 15′	-	9·9811711
	To op. ſide AC	306·19	-	2·4859949

EXAM. VI. From the edge of a ditch, of 36 feet wide, ſurrounding a fort, having taken the angle of elevation of the top of the wall, it was found to be 62° 40′: required the height of the wall, and the length of a ladder to reach from my ſtation to the top of it?

Anſ. { height of wall 69·64, ladder 78·4 feet.

EXAM. VII. Required the length of a ſhoar, which, being to ſtrut 11 feet from the upright of a building, will ſupport a jamb 23 feet 10 inches from the ground?

Anſ. 26 feet 3 inches.

EXAM. VIII. A ladder, 40 feet long, can be ſo planted, that it ſhall reach a window 33 feet from the ground, on one ſide of the ſtreet; and by turning it over, without moving the foot out of its place, it will do the ſame by a window 21 feet high, on the other ſide: required the breadth of the ſtreet? Anſ. 56·649 feet.

EXAM. IX. A Maypole, whoſe top was broken off by a blaſt of wind, ſtruck the ground at 15 feet diſtance from the foot of the pole: what was the height of the whole maypole, ſuppoſing the broken piece to meaſure 39 feet in length?

Anſ. 75 feet.

EXAM. X. At 170 feet diſtance from the bottom of a tower, the angle of its elevation was found to be 52° 30′: required the altitude of the tower? Anſ. 221 feet.

EXAM. XI. From the top of a tower, by the ſea-ſide, of 143 feet height, it was obſerved that the angle of depreſſion of a ſhip's bottom, then at anchor, meaſured 35°; what then was the ſhip's diſtance from the bottom of the wall?

Anſ. 204·22 feet.

EXAM. XII. What is the perpendicular height of a hill; its angle of elevation, taken at the bottom of it, being 46°, and

and 200 yards farther off, on a level with the bottom of it, the angle was 31° ? Anſ. 286·28 yards.

EXAM. XIII. Wanting to know the height of an inacceſſible tower; at the leaſt diſtance from it, on the ſame horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°: required its height, and my diſtance from it at the firſt ſtation ?

Anſ. { height 307·53
diſtance 192·15

EXAM. XIV. Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inacceſſible hill; I took the angle of elevation of the top of the hill equal 40°, and of the top of the tower equal 51°; then meaſuring in a line directly from it to the diſtance of 200 feet farther, I found the angle to the top of the tower to be 33° 45′. What then is the height of the tower ?

Anſ. 93·33148 feet.

EXAM. XV. From a window near the bottom of a houſe, which ſeemed to be on a level with the bottom of a ſteeple, I took the angle of elevation of the top of the ſteeple equal 40°; then from another window, 18 feet directly above the former, the like angle was 37° 30′: what then is the height and diſtance of the ſteeple ?

Anſ. { height 210·44
diſtance 250·79

EXAM. XVI. Wanting to know the height of, and my diſtance from, an object on the other ſide of a river, which ſeemed to be on a level with the place where I ſtood, cloſe by the ſide of the river; and not having room to meaſure backward, on the ſame plane, becauſe of the immediate riſe of the bank, I placed a mark where I ſtood, and meaſured in a direction from the object, up the aſcending ground to the diſtance of 264 feet, where it was evident that I was above the level of the top of the object; there the angles of depreſſion were found to be, viz. of the mark left at the river's ſide 42°, of the bottom of the object 27°, and of its top 19°. Required then the height of the object, and the diſtance of the mark from its bottom ?

Anſ. { height 57·26
diſtance 159·50

EXAM. XVII. If the height of the mountain called the Pike of Teneriff be 3 miles, and the angle taken at the top of it, as formed between a plumb line and a line conceived

to

to touch the earth in the horizon, or fartheſt viſible point, be 87° 46′ 33″; it is required from hence to determine the magnitude of the whole earth, and the utmoſt diſtance that can be ſeen on its ſurface from the top of the mountain, ſuppoſing the form of the earth to be perfectly round?

Anſ. { diſt. 154·539 ; diam. 7957·75 } miles.

Exam. xviii. Two ſhips of war, intending to cannonade a fort, are, by the ſhallowneſs of the water, kept ſo far from it, that they ſuſpect their guns cannot reach it with effect. In order, therefore, to meaſure the diſtance, they ſeparate from each other a quarter of a mile, or 440 yards; then each ſhip obſerves and meaſures the angle which the other ſhip and the fort ſubtends, which angle were 83° 45′ and 85° 15′. What then was the diſtance between each ſhip and the fort?

Anſ. { 2292·26 yards. ; 2298·05 }

Exam. xix. Being on the ſide of a river, and wanting to know the diſtance to a houſe which was ſeen at a diſtance on the other ſide; I meaſured out for a baſe 400 yards in a right line by the ſide of the river, and found that the two angles, one at each end of this line, ſubtended by the other end and the houſe, were 68° 2′ and 73° 15′. What then was the diſtance between each ſtation and the houſe?

Anſ. { 593·08 yards. ; 612·38 }

Exam. xx. Wanting to know the breadth of a river, I meaſured a baſe of 500 yards in a ſtraight line cloſe by one ſide of it; and at each end of this line I found the angles ſubtended by the other end and a tree cloſe on the bank on the other ſide of the river, to be 53° and 79° 12′. What then was the perpendicular breadth of the river?

Anſ. 529·48 yards.

Exam. xxi. Wanting to know the extent of a piece of water, or diſtance between two headlands; I meaſured from each of them to a certain point inland, and found the two diſtances to be 735 yards and 846 yards; alſo, the horizontal angle ſubtended between theſe two lines was 55° 40′. What then was the diſtance required? Anſ. 741·2 yards.

Exam. xxii. A point of land was obſerved, by a ſhip at ſea, to bear eaſt-by-ſouth; and after ſailing north-eaſt 12 miles, it was found to bear ſouth-eaſt-by-eaſt. It is required to

to determine the place of that headland, and the ſhip's diſtance from it at the laſt obſervation. Anſ. 26·0728 miles.

Exam. xxiii. Wanting to know the diſtance between a houſe and a mill, which were ſeen at a diſtance on the other ſide of a river, I meaſured a baſe line along the ſide, where I was, of 600 yards, and at each end of it took the angles ſubtended by the other end and the houſe and mill, which were as follow, viz. at one end the angles were 58° 20′ and 95° 20′, and at the other end the like angles were 53° 30′ and 98° 45′. What then was the diſtance between the houſe and mill?

Anſ. 959·5866 yards.

Exam. xxiv. Wanting to know my diſtance from an inacceſſible object O, on the other ſide of a river; and having no inſtrument for taking angles, but only a chain or cord for meaſuring diſtances; from each of two ſtations, A and B, which were taken at 500 yards aſunder, I meaſured in a direct line from the object O 100 yards, viz. AC and BD each equal to 100 yards; alſo the diagonal AD meaſured 550 yards, and the diagonal BC 560. What then was the diſtance of the object O from each ſtation A and B?

Anſ. { AO 536·25
BO 500·09

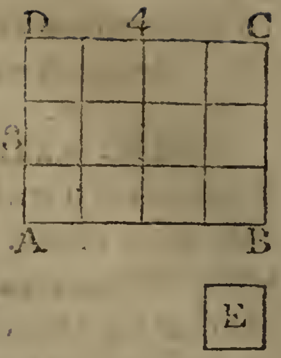
MENSURATION OF PLANES.

THE Area of any plane figure, is the meaſure of the ſpace contained within its extremes or bounds; without any regard to thickneſs.

This area, or the content of the plane figure, is eſtimated by the number of little ſquares that may be contained in it; the ſide of thoſe little meaſuring ſquares being an inch, a foot, a yard, or any other fixed quantity. And hence, the area or content is ſaid to be ſo many ſquare inches, or ſquare feet, or ſquare yards, &c.

Thus,

Thus, if the figure to be meaſured be the rectangle ABCD, and the little ſquare E, whoſe ſide is one inch, be the meaſuring unit propoſed: then, as often as the ſaid little ſquare is contained in the rectangle, ſo many ſquare inches the rectangle is ſaid to contain, which in the preſent caſe is 12.



PROBLEM I.

To find the Area of any Parallelogram, whether it be a Square, a Rectangle, a Rhombus, or a Rhomboid.

MULTIPLY the length by the perpendicular breadth, or height, and the product will be the area*.

EXAMPLES.

Ex. 1. To find the area of a parallelogram, whoſe length is 12·25, and height 8·5.

12·25	length
8·5	breadth
6125	
9800	
104·125	area

* The truth of this rule is proved in the Geom. theor. 81, cor. 2.

The ſame is otherwiſe proved thus: Let the foregoing rectangle be the figure propoſed; and let the length and breadth be divided into equal parts, each equal to the linear meaſuring unit, being here 4 for the length, and 3 for the breadth; and let the oppoſite points of diviſion be connected by right lines.—Then, it is evident that theſe lines divide the rectangle into a number of little ſquares, each equal to the ſquare meaſuring unit E; and farther, that the number of theſe little ſquares, or the area of the figure, is equal to the number of linear meaſuring units in the length, repeated as often as there are linear meaſuring units in the breadth, or height; that is, equal to the length drawn into the height; which here is 4 × 3 or 12.

And it is proved (Geom. theor. 25, cor. 2), that a rectangle is equal to any oblique parallelogram, of equal length and perpendicular breadth. Therefore the rule is general for all parallelograms whatever.

Ex. 2.

Ex. 2. To find the area of a ſquare, whoſe ſide is 35·25 chains. Anſ. 124 acres, 1 rood, 1 perch.

Ex. 3. To find the area of a rectangular board, whoſe length is $12\frac{1}{2}$ feet, and breadth 9 inches. Anſ. $9\frac{3}{8}$ feet.

Ex. 4. To find the content of a piece of land, in form of a rhombus, its length being 6·20 chains, and perpendicular height 5·45. Anſ. 3 acres, 1 rood, 20 perches.

Ex. 5. To find the number of ſquare yards of painting in a rhomboid, whoſe length is 37 feet, and breadth 5 feet 3 inches. Anſ. $21\frac{7}{12}$ ſquare yards.

PROBLEM II.

To find the Area of a Triangle.

Rule I. Multiply the baſe by the perpendicular height, and take half the product for the area*. Or, multiply the one of theſe dimenſions by half the other.

EXAMPLES.

Ex. 1. To find the area of a triangle, whoſe baſe is 625, and perpendicular height 520 links?

Here $625 \times 260 = 162500$ ſquare links,
or equal 1 acre, 2 roods, 20 perches, the anſwer.

Ex. 2. How many ſquare yards contains the triangle, whoſe baſe is 40, and perpendicular 30 feet?
Anſ. $66\frac{2}{3}$ ſquare yards.

Ex. 3. To find the number of ſquare yards in a triangle, whoſe baſe is 49 feet, and height $25\frac{1}{4}$ feet?
Anſ. $68\frac{53}{72}$, or 68·7361.

Ex. 4. To find the area of a triangle, whoſe baſe is 18 feet 4 inches, and height 11 feet 10 inches?
Anſ. 108 feet, $5\frac{2}{3}$ inches.

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* The truth of this rule is evident, becauſe any triangle is the half of a parallelogram of equal baſe and altitude, by Geom. theor. 26.

Rule II.

RULE II. When two ſides and their contained angle are given: Multiply the two given ſides together, and take half their product: Then ſay, as radius is to the ſine of the given angle, ſo is that half product, to the area of the triangle.

Or, multiply that half product by the ſine of the ſaid angle*.

Ex. 1. What is the area of a triangle, whoſe two ſides are 30 and 40, and their contained angle 28° 57′ 18″?

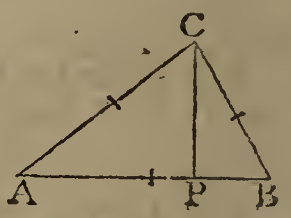
Here $\frac{1}{2} \times 40 \times 30 = 600$,
therefore 1 : ·4841226 nat. ſin. 28° 57′ 18″
600

290·47356 the anſwer.

Ex. 2. How many ſquare yards contains the triangle, of which one angle is 45°, and its containing ſides 25 and $21\frac{1}{4}$ feet? Anſ. 20·86947.

RULE III. When the three ſides are given: Add all the three ſides together, and take half that ſum. Next, ſubtract each ſide ſeverally from the ſaid half ſum, obtaining three remainders. Then multiply the ſaid half ſum and thoſe three remainders all together, and, laſtly, extract the ſquare root of the laſt product, for the area of the triangle †.

Ex. 1.



* For, let AB, AC be the two given ſides, including the given angle A. Now $\frac{1}{2}AB \times CP$ is the area, by the firſt rule, CP being perpendicular. But, by trigon. as ſin. ∠ P, or radius : AC :: ſin. ∠ A : CP = ſin. ∠ A × AC, taking radius = 1. Therefore the area $\frac{1}{2}AB \times CP$ is $= \frac{1}{2}AB \times AC \times$ ſin. ∠ A, to radius 1; or as radius : ſin. ∠ A :: $\frac{1}{2}AB \times AC$: the area.

† For, let b denote the baſe AB of the triangle ABC (ſee the laſt fig.), alſo a the ſide AC, and c the ſide BC. Then, by th. 3, Trigon. as $b : a + c :: a - c : \frac{aa - cc}{b} =$ AP — PB the diff. of the ſegments;

theref.

Ex. 1. To find the area of the triangle whose three sides are 20, 30, 40.

20	45	45	45
30	20	30	40
40	—	—	—
——	25 1st rem.	15 2d rem.	5 3d rem.
2) 90	—	—	—
45 half sum			

Then $45 \times 25 \times 15 \times 5 = 84375$.
The root of which is 290·4737, the area.

Ex. 2. How many square yards of plastering are in a triangle, whose sides are 30, 40, 50 feet? Ans. $66\frac{2}{3}$.

Ex. 3. How many acres, &c. contains the triangle, whose sides are 2569, 4900, 5025 links?

Ans. 61 acres, 1 rood, 39 perches.

PROBLEM III.

To find the Area of a Trapezoid.

ADD together the two parallel sides; then multiply their sum by the perpendicular breadth, or the distance between them; and take half the product for the area. By Geom. theor. 29.

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theref. $\frac{1}{2}b + \frac{aa - cc}{2b} = \frac{bb + aa - cc}{2b}$ = the segment AP;

hence $\sqrt{AC^2 - AP^2}$ = the perp. CP, that is,

$\sqrt{(aa - (\frac{bb + aa - cc}{2b})^2)}$ = - - - - - - - - - - -

$\sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{4bb}}$ = CP.

But $\frac{1}{2}AB \times CP$ is the area, that is,

$\frac{1}{2}b \times CP = \sqrt{\frac{2a^2b^2 - a^4 + 2b^2c^2 - b^4 + 2a^2c^2 - c^4}{16}}$

$= \sqrt{(\frac{-aa + bb + cc + 2bc}{4} \times \frac{aa - bb - cc + 2bc}{4})}$

$= \sqrt{(\frac{a + b + c}{2} \times \frac{-a + b + c}{2} \times \frac{a - b + c}{2} \times \frac{a + b - c}{2})}$

$= \sqrt{(s \times \overline{s - a} \times \overline{s - b} \times \overline{s - c})}$, which is the rule, where s denotes half the sum of the three sides.

Ex. 1.

Ex. 1. In a trapezoid, the parallel ſides are 750 and 1225, and the perpendicular diſtance between them 1540 links: to find the area.

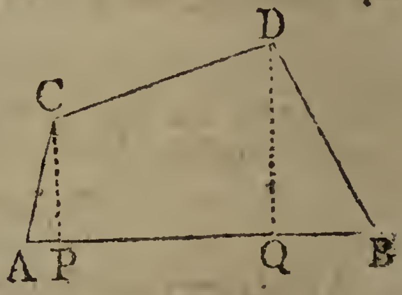
$$\begin{array}{r} 1225 \\ 750 \\ \hline 1975 \times 770 = 152075 \end{array}$$ ſquare links = 15 acr. 33 perc.

Ex. 2. How many ſquare feet are contained in the plank, whoſe length is 12 feet 6 inches, the breadth at the greater end 15 inches, and at the leſs end 11 inches?

Anſ. $13\frac{13}{24}$ feet.

Ex. 3. In meaſuring along one ſide AB of a quadrangular field, that ſide and the two perpendiculars let fall on it from the two oppoſite corners, meaſured as below: required the content.

AP = 110 links.
AQ = 745
AB = 1110
CP = 352
DQ = 595



Anſ. 4 acres, 1 rood, 5·792 perches.

PROBLEM IV.

To find the Area of any Trapezium.

Divide the trapezium into two triangles by a diagonal; then find the areas of theſe triangles, and add them together.

Note. If two perpendiculars be let fall on the diagonal, from the other two oppoſite angles, the ſum of theſe perpendiculars being multiplied by the diagonal, half the product will be the area of the trapezium.

Ex. 1. To find the area of the trapezium, whoſe diagonal is 42, and the two perpendiculars on it 16 and 18.

Here 16 + 18 = 34, its half is 17.
Then 42 × 17 = 714 the area.

Ex. 2. How many ſquare yards of paving are in the trapezium, whoſe diagonal is 65 feet, and the two perpendiculars let fall on it 28 and $33\frac{1}{2}$ feet? Anſ. $222\frac{1}{12}$ yards.

Ex. 3.

Ex. 3. In the quadrangular field ABCD, on account of obstructions there could only be taken the following measures, viz. the two sides BC 265 and AD 220 yards, the diagonal AC 378, and the two distances of the perpendiculars from the ends of the diagonal, namely, AE 100, and CF 70 yards. Required the area in acres, when 4840 square yards make an acre? Ans. 17 acres, 2 roods, 21 perches.

PROBLEM V.

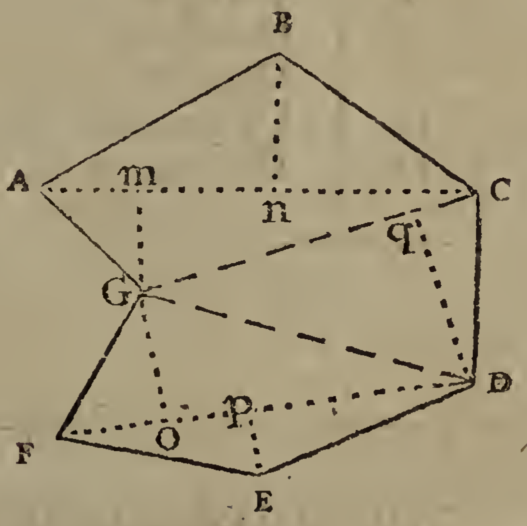
To find the Area of an Irregular Polygon.

DRAW diagonals dividing the proposed polygon into trapeziums and triangles. Then find the areas of all these separately, and add them together for the content of the whole polygon.

EXAM. To find the content of the irregular figure ABC DEFGA, in which are given the following diagonals and perpendiculars: namely,

AC	55
FD	52
GC	44
Gm	13
Bn	18
Go	12
Ep	8
Dq	23

Ans. $1878\frac{1}{2}$



PROBLEM VI.

To find the Area of a Regular Polygon.

RULE I. MULTIPLY the perimeter of the polygon, or sum of its sides, by the perpendicular drawn from its centre on one of its sides, and take half the product for the area*.

* This is only in effect resolving the polygon into as many equal triangles as it has sides, by drawing lines from the centre to all the angles; then finding their areas, and adding them all together.

Ex. 1.

Ex. 1. To find the area of the regular pentagon, each side being 25 feet, and the perpendicular from the centre on each side is 17·2047737.

Here $25 \times 5 = 125$ is the perimeter.
And $17{\cdot}2047737 \times 125 = 2150{\cdot}5967125$.
Its half 1075·298356 is the area sought.

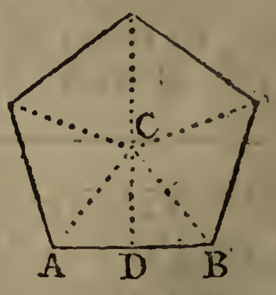
Rule ii. Square the side of the polygon; then multiply that square by the area or multiplier set against its name in the following table, and the product will be the area*.

No. of Sides.	Names.	Areas, or Multipliers.
3	Trigon, or triangle	0·4330127
4	Tetragon, or square	1·0000000
5	Pentagon	1·7204774
6	Hexagon	2·5980762
7	Heptagon	3·6339124
8	Octagon	4·8284271
9	Nonagon	6·1818242
10	Decagon	7·6942088
11	Undecagon	9·3656399
12	Dodecagon	11·1961524

Exam-

* This rule is founded on the property, that like polygons, being similar figures, are to one another as the squares of their like sides; which is proved in the Geom. theor. 89. Now, the multipliers in the table, are the areas of the respective polygons to the side 1. Whence the rule is manifest.

Note. The areas in the table, to each side 1, may be computed in the following manner: From the centre C of the polygon draw lines to every angle, dividing the whole figure into as many equal triangles as the polygon has sides; and let ABC be one of those triangles, the perpendicular of which is CD. Divide 360 degrees by the number of sides in the polygon, the quotient gives the angle at the centre ACB. The half of



EXAM. Taking here the ſame example as before, namely a pentagon, whoſe ſide is 25 feet.

Then 25^2 being $= 625$,
And the tabular area 1·7204774;
Theref. $1·7204774 \times 625 = 1075·298375$, as before.

Ex. 2. To find the area of the trigon, or equilateral triangle, whoſe ſide is 20. Anſ. 173·20508.

Ex. 3. To find the area of the hexagon, whoſe ſide is 20. Anſ. 1039·23048.

Ex. 4. To find the area of an octagon, whoſe ſide is 20. Anſ. 1931·37084.

Ex. 5. To find the area of a decagon, whoſe ſide is 20. Anſ. 3077·68352.

PROBLEM VII.

To find the Diameter and Circumference of any Circle, the one from the other.

THIS may be done nearly by either of the two following proportions, viz.

As 7 is to 22, ſo is the diameter to the circumference.
Or, As 1 is to 3·1416, ſo is the diameter to the circumference*.

Ex. 1.

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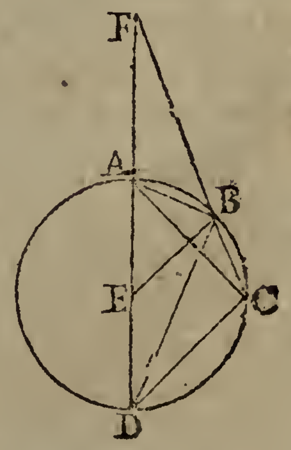
of this gives the angle ACD; and this taken from 90°, leaves the angle CAD. Then, as radius is to AD, ſo is tang. angle CAD, to the perpendicular CD. This multiplied by AD, gives the area of the triangle ABC; which being multiplied by the number of the triangles, or of the ſides of the polygon, gives its whole area, as in the table.



* For, let ABCD be any circle, whoſe centre is E, and let AB, BC be any two equal arcs. Draw the ſeveral chords as in the figure, and join BE; alſo draw the diameter DA, which produce to F, till BF be equal to the chord BD.

Then the two iſoſceles triangles DEB, DBF are equiangular, becauſe they have the angle at D common; conſequently DE : DB :: DB : DF. But the two triangles AFB, DCB are identical, or equal in all reſpects, becauſe they have the angle F = the angle BDC, being each equal to the angle ADB, theſe being ſubtended by the equal arcs

Ex. 1. To find the circumference of the circle whose diameter is 20.

By the first rule, as $7 : 22 :: 20 : 62\frac{6}{7}$, the answer.

Ex. 2.

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arcs AB, BC; also the exterior angle FAB of the quadrangle ABCD, is equal to the opposite interior angle at C; and the two triangles have also the side BF = the side BD; therefore the side AF is also equal to the side DC. Hence the proportion above, viz. $DE : DB :: DB : DF = DA + AF$, becomes $DE : DB :: DB : 2DE + DC$. Then, by taking the rectangles of the extremes and means, it is $DB^2 = 2DE^2 + DE \cdot DC$.

Now, if the radius DE be taken = 1, this expression becomes $DB^2 = 2 + DC$, and hence the root $DB = \sqrt{2 + DC}$. That is, If the measure of the supplemental chord of any arc be increased by the number 2, the square root of the sum will be the supplemental chord of half that arc.

Now, to apply this to the calculation of the circumference of the circle, let the arc AC be taken equal to $\frac{1}{6}$ of the circumference, and be successively bisected by the above theorem: thus, the chord AC, of $\frac{1}{6}$ of the circumference, is the side of the inscribed regular hexagon, and is therefore equal the radius AE or 1: hence, in the right-angle triangle ACD, it will be $DC = \sqrt{AD^2 - AC^2} = \sqrt{2^2 - 1^2} = \sqrt{3} = 1{\cdot}7320508076$, the supplemental chord of $\frac{1}{6}$ of the periphery.

Then, by the foregoing theorem, by always bisecting the arcs, and adding 2 to the last square root, there will be found the supplemental chords of the 12th, the 24th, the 48th, the 96th, &c. parts of the periphery; thus,

$$\left.\begin{array}{lcl}
\sqrt{3{\cdot}7320508076} & = & 1{\cdot}9318516525 \\
\sqrt{3{\cdot}9318516525} & = & 1{\cdot}9828897227 \\
\sqrt{3{\cdot}9828897227} & = & 1{\cdot}9957178465 \\
\sqrt{3{\cdot}9957178465} & = & 1{\cdot}9989291743 \\
\sqrt{3{\cdot}9989291743} & = & 1{\cdot}9997322757 \\
\sqrt{3{\cdot}9997322757} & = & 1{\cdot}9999330678 \\
\sqrt{3{\cdot}9999330678} & = & 1{\cdot}9999832669 \\
\sqrt{3{\cdot}9999832669} & = & \text{- - - -}
\end{array}\right\}
\begin{array}{c}\text{for the supplemental}\\ \text{chord of}\end{array}
\left\{\begin{array}{c}
\frac{1}{12} \\ \frac{1}{24} \\ \frac{1}{48} \\ \frac{1}{96} \\ \frac{1}{192} \\ \frac{1}{384} \\ \frac{1}{768} \\ \frac{1}{1536}
\end{array}\right\}
\text{of the periphery.}$$

Since then it is found that $3{\cdot}9999832669$ is the square of the supplemental chord of the 1536th part of the periphery, let this number be taken from 4, which is the square of the diameter, and the remainder $0{\cdot}0000167331$ will be the square of the chord of the said 1536th part of the periphery, and consequently the root $\sqrt{0{\cdot}0000167331} = 0{\cdot}0040906112$ is the length of that chord; this number then being multiplied by 1536, gives $6{\cdot}2831788$ for the perimeter of a regular polygon of 1536 sides inscribed in the circle; which, as the sides

Ex. 2. If the circumference of the earth be 25000 miles, what is its diameter?

By the 2d rule, as 3·1416 : 1 :: 25000 : 7957¾ nearly the diameter.

PROBLEM VIII.

To find the Length of any Arc of a Circle.

MULTIPLY the degrees in the given arc by the radius of the circle, and the product again by the decimal ·01745, for the length of the arc *.

Ex. 1.

ſides of the polygon nearly coincide with the circumference of the circle, muſt alſo expreſs the length of the circumference itſelf, very nearly.

But now, to ſhew how near this determination is to the truth, let AQP = 0·0040906112 repreſent one ſide of ſuch a regular polygon of 1536 ſides, and SRT a ſide of another ſimilar polygon deſcribed about the circle; and from the centre E let the perpendicular EQR be drawn, biſecting AP and ST in Q and R. Then, ſince AQ is = ½AP = - - 0.0020453056, and EA = 1, therefore $EQ^2 = EA^2 - AQ^2 =$ ·9999958167, and conſequently its root gives EQ = ·9999979084; then, becauſe of the parallels AP, ST, it is EQ : ER :: AP : ST :: as the whoſe inſcribed perimeter : to the circumſcribed one, that is, as ·9999979084 : 1 :: 6·2831788 : 6·2831920 the perimeter of the circumſcribed polygon. Now, the circumference of the circle being greater than the perimeter of the inner polygon, but leſs than that of the outer, it muſt conſequently be greater than 6·2831788, but leſs than 6·2831920, and muſt therefore be nearly equal ½ their ſum, or 6·2831854, which in fact is true to the laſt figure, which ſhould be a 3 inſtead of the 4.

Hence, the circumference being 6·2831854 when the diameter is 2, it will be the half of that, or 3·1415927, when the diameter is 1, to which the ratio in the rule, viz. 1 to 3·1416 is very near. Alſo, the other ratio in the rule 7 to 22 or 1 to $3\frac{1}{7}$ = 3·1428 &c. is another near approximation.

* It having been found, in the demonſtration of the foregoing problem, that when the radius of a circle is 1, the length of the whole circumference is 6·2831854, which conſiſts of 360 degrees; therefore as 360° : 6·2831854 :: 1° : ·01745 &c. the length of the arc of 1 degree. Hence, the number ·01745 multiplied by any number of degrees, will give the length of the arc of thoſe degrees.

 And,

Ex. 1. To find the length of an arc of 30 degrees, the radius being 9 feet. Anſ. 4·7115.

Ex. 2. To find the length of an arc of 12° 10′, or $12°\frac{1}{6}$, the radius being 10 feet. Anſ. 2·1231.

PROBLEM IX.

To find the Area of a Circle.*

Rule I. Multiply half the circumference by half the diameter. Or multiply the whole circumference by the whole diameter, and take $\frac{1}{4}$ of the product.

Rule II. Square the diameter, and multiply that ſquare by the decimal ·7854, for the area.

Rule III. Square the circumference, and multiply that ſquare by the decimal ·07958.

Ex. 1. To find the area of a circle whoſe diameter is 10, and its circumference 31·416.

By Rule 1.	By Rule 2.	By Rule 3.
31·416	·7854	31·416
10	$10^2 =$ 100	31·416
4) 314·16	78·54	986·965
78·54		·07958
		78·54

So that the area is 78·54 by all the rules.

And, becauſe the circumferences, and arcs, are as the diameters, or as the radii of the circles, therefore as the radius 1 is to any other radius r, ſo is the length of the arc above mentioned, to $r \times$ ·01745 $\times$ degrees in the arc, which is the length of that arc, as in the rule.

* The firſt rule is proved in the Geom. theor. 94.

And the 2d and 3d rules are deduced from the firſt rule, in this manner.—By that rule, $dc \div 4$ is the area, when d denotes the diameter, and c the circumference. But, by prob. 7, c is $= 3·1416d$; therefore the ſaid area $dc \div 4$, becomes $d \times 3·1416\,d \div 4 =$ $·7854\,d^2$, which gives the 2d rule.—Alſo, by the ſame prob. 7, d is $= c \div 3·1416$; therefore again the ſame firſt area $dc \div 4$, becomes $c \div 3·1416 \times c \div 4 = c^2 \div 12·5664$, which is $= c^2 \times$ ·07958, by taking the reciprocal of 12·5664, or changing that diviſor into the multiplier ·07958; which gives the 3d rule.

Corol. Hence, the areas of different circles are in proportion to one another, as the ſquare of their diameters, or as the ſquare of their circumferences; as before proved in the Geom. theor. 93.

Ex. 2. To find the area of a circle, whose diameter is 7, and circumference 22. Anf. $38\frac{1}{2}$.

Ex. 3. How many square yards are in a circle, whose diameter is $3\frac{1}{2}$ feet? Anf. 1·069.

Ex. 4. To find the area of a circle whose circumference is 12 feet. Anf. 11·4595.

PROBLEM X.

To find the Area of a Circular Ring, or Space included between two Concentric Circles.

Take the difference between the areas of the two circles, as found by the laſt problem, for the area of the ring.—Or, which is the ſame thing, ſubtract the ſquare of the leſs diameter from the ſquare of the greater, and multiply their difference by ·7854.—Or laſtly, multiply the ſum of the diameters by the difference of the ſame, and that product by ·7854; which is ſtill the ſame thing, becauſe the product of the ſum and difference of any two quantities, is equal to the difference of their ſquares.

Ex. 1. The diameters of two concentric circles being 10 and 6, required the area of the ring contained between their circumferences.

Here $10 + 6 = 16$ the ſum, and $10 - 6 = 4$ the diff.
Therefore $\cdot 7854 \times 16 \times 4 = \cdot 7854 \times 64 = 50\cdot 2656$, the area.

Ex. 2. What is the area of the ring, the diameters of whoſe bounding circles are 10 and 20? Anf. 235·62.

PROBLEM XI.

To find the Area of the Sector of a Circle.

Rule I. Multiply the radius, or half the diameter, by half the arc of the ſector, for the area. Or, multiply the whole diameter by the whole arc of the ſector, and take $\frac{1}{4}$ of the product. The reaſon of which is the ſame as for the firſt rule to problem 9.

Rule II. As 360 is to the degrees in the arc of the ſector, ſo is the area of the whole circle, to the area of the ſector.

This is evident, becauſe the ſector is proportional to the length of the arc, or to the degrees contained in it.

Ex. 1. To find the area of a circular ſector, whoſe arc contains 18 degrees; the diameter being 3 feet?

1. By

1. By the 1ſt Rule.

Firſt, $3{\cdot}1416 \times 3 = 9{\cdot}4248$, the circumference.
And $360 : 18 :: 9{\cdot}4248 : {\cdot}47124$, the length of the arc.
Then ${\cdot}47124 \times 3 \div 4 = {\cdot}11781 \times 3 = {\cdot}35343$, the area.

2. By the 2d Rule.

Firſt, ${\cdot}7854 \times 3^2 = 7{\cdot}0686$, the area of the whole circle.
Then, as $360 : 18 :: 7{\cdot}0686 : {\cdot}35343$, the area of the ſector.

Ex. 2. To find the area of a ſector, whoſe radius is 10, and arc 20. Anſ. 100.

Ex. 3. Required the area of a ſector, whoſe radius is 25, and its arc containing 147° 29′. Anſ. 804·3986.

PROBLEM XII.

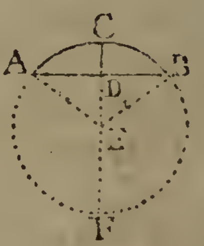
To find the Area of a Segment of a Circle.

Rule 1. Find the area of the ſector having the ſame arc with the ſegment, by the laſt problem.

Find alſo the area of the triangle, formed by the chord of the ſegment and the two radii of the ſector.

Then take the ſum of theſe two for the anſwer, when the ſegment is greater than a ſemicircle: or take their difference for the anſwer, when it is leſs than a ſemicircle.—As is evident by inſpection.

Ex. 1. To find the area of the ſegment ACBDA, its chord AB being 12, and the radius AE or CE 10.



Firſt, As AE : AD :: ſin. ∠D90° : ſin. $36° \; 52'\tfrac{1}{5} = 36{\cdot}87$ degrees, the degrees in the ∠AEC or arc AC. Their double, 73·74, are the degrees in the whole arc ACB.

Now ${\cdot}7854 \times 400 = 314{\cdot}16$, the area of the whole circle,

Therefore $360° : 73{\cdot}74 :: 314{\cdot}16 : 64{\cdot}3504$, area of the ſector ACBE.

Again, $\sqrt{AE^2 - AD^2} = \sqrt{100 - 36} = \sqrt{64} = 8 = DE$.
Theref. $AD \times DE = 6 \times 8 = 48$, the area of the triangle AEB.
Hence, ſector ACBA — triangle AEB $= 16{\cdot}3504$, area of ſeg. ACBDA.

Rule 1.

RULE II. Divide the height of the ſegment by the diameter; and find the quotient in the column of heights in the following tablet:—Take out the correſponding area in the next column on the right hand; and multiply it by the ſquare of the circle's diameter, for the area of the ſegment *.

Note. When the quotient is not found exactly in the table, proportion may be made between the next leſs and greater area, in the ſame manner as is done for logarithms, or any other table.

Table of the Areas of Circular Segments.

Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.	Height.	Area of the Segm.
·01	·00133	·11	·04701	·21	·11990	·31	·20738	·41	·30319
·02	·00375	·12	·05339	·22	·12811	·32	·21667	·42	·31304
·03	·00687	·13	·06000	·23	·13646	·33	·22603	·43	·32293
·04	·01054	·14	·06683	·24	·14494	·34	·23547	·44	·33284
·05	·01468	·15	·07387	·25	·15354	·35	·24498	·45	·34278
·06	·01924	·16	·08111	·26	·16226	·36	·25455	·46	·35274
·07	·02417	·17	·08853	·27	·17109	·37	·26418	·47	·36272
·08	·02944	·18	·09613	·28	·18002	·38	·27386	·48	·37270
·09	·03502	·19	·10390	·29	·18905	·39	·28359	·49	·38270
·10	·04088	·20	·11182	·30	·19817	·40	·29337	·50	·39270

Ex. 2. Taking the ſame example as before, in which are given the chord AB 12, and the radius 10, or diameter 20.

And having found, as above, DE = 8; then CE — DE = CD = 10 — 8 = 2. Hence, by the rule, CD ÷ CF = 2 ÷ 20 = ·1 the tabular height. This being found in the firſt column of the table, the correſponding tabular area is ·04088. Then ·04088 × 20^2 = ·04088 × 400 = 16·352, the area, nearly the ſame as before.

* The truth of this rule depends on the principle of ſimilar plane figures, which are to one another as the ſquare of their like linear dimenſions. The ſegments in the table are thoſe of a circle whoſe diameter is 1; and the firſt column contains the correſponding heights or verſed ſines divided by the diameter. Thus then, the area of the ſimilar ſegment, taken from the table, and multiplied by the ſquare of the diameter, gives the area of the ſegment to this diameter.

Ex. 3. What is the area of the ſegment, whoſe height is 18, and diameter of the circle 50? Anſ. 636·375.

Ex. 4. Required the area of the ſegment whoſe chord is 16, the diameter being 20? Anſ. 44·728.

PROBLEM XIII.

To meaſure long Irregular Figures.

* TAKE or meaſure the breadth in ſeveral places, at equal diſtances. Then add all theſe breadths together, and divide the ſum by the number of them, for the mean breadth; which multiply by the length, for the area.

Note 1. Take half the ſum of the extreme breadths for one of the ſaid breadths; which makes the number to divide by the ſame as the number of parts in the baſe.

Note 2. If the perpendiculars or breadths be not at equal diſtances, compute all the parts ſeparately, as ſo many trapezoids, and add them all together for the whole area.

Or elſe, add all the perpendicular breadths together, and divide their ſum by the number of them for the mean breadth, to multiply by the length; which will give the whole area, not far from the truth.

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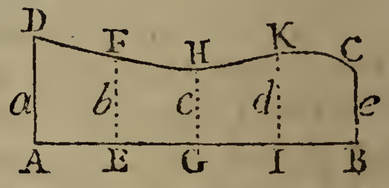
* This rule is made out as follows:— Let ABCD be the irregular piece; having the ſeveral breadths AD, EF, GH, IK, BC, at the equal diſtances AE, EG, GI, IB. Let the ſeveral breadths in order be denoted by the correſponding letters a, b, c, d, e, and the whole length AB by l; then compute the areas of the parts into which the figure is divided by the perpendiculars, as ſo many trapezoids, by prob. 3, and add them all together. Thus, the ſum of the parts is,



$$\frac{a+b}{2} \times AE + \frac{b+c}{2} \times EG + \frac{c+d}{2} \times GI + \frac{d+e}{2} \times IB$$

$$= \frac{a+b}{2} \times \tfrac{1}{4}l + \frac{b+c}{2} \times \tfrac{1}{4}l + \frac{c+d}{2} \times \tfrac{1}{4}l + \frac{d+e}{2} \times \tfrac{1}{4}l$$

$$= (\tfrac{1}{2}a + b + c + d + \tfrac{1}{2}e) \times \tfrac{1}{4}l = (m + b + c + d)\,\tfrac{1}{4}l,$$

which is the whole area, agreeing with the rule; m being the arithmetic mean between the extremes, or half the ſum of them both, and 4 the number of the parts. And the ſame for any other number of parts.

Ex. 1. The breadths of an irregular figure, at five equidiſtant places, being 8·2, 7·4, 9·2, 10·2, 8·6; and the whole length 39; required the area?

Firſt, (8·2 + 8·6) ÷ 2 = 8·4, the mean of the two extremes.
Then 8·4 + 7·4 + 9·2 + 10·2 = 35·2, ſum of breadths.
And 35·2 ÷ 4 = 8·8, the mean breadth.
Hence 8·8 × 39 = 343·2, the anſwer.

Ex. 2. The length of an irregular figure being 84, and the breadths at ſix equidiſtant places 17·4, 20·6, 14·2, 16·5, 20·1, 24·4; what is the area? Anſ. 1550·64.

MENSURATION OF SOLIDS.

By the Menſuration of Solids are determined the ſpaces included by contiguous ſurfaces, and the ſum of the meaſures of theſe including ſurfaces, is the whole ſurface or ſuperficies of the body.

The meaſure of a ſolid, is called its ſolidity, capacity, or content.

Solids are meaſured by cubes, whoſe ſides are inches, or feet, or yards, &c. And hence the ſolidity of a body is ſaid to be ſo many cubic inches, feet, yards, &c. as will fill its capacity or ſpace, or another of equal magnitude.

The leaſt ſolid meaſure is the cubic inch, other cubes being taken from it according to the proportion in the following table.

Table of Cubic or Solid Meaſures.

1728	cubic inches make	1 cubic foot
27	cubic feet -	1 cubic yard
166⅜	cubic yards -	1 cubic pole
64000	cubic poles -	1 cubic furlong
512	cubic furlongs	1 cubic mile.

PRO-

PROBLEM I.

To find the Superficies of a Prism.

MULTIPLY the perimeter of one end of the prism by the length or height of the solid, and the product will be the surface of all its sides. To which add also the area of the two ends of the prism, when required*.

Or, compute the areas of all the sides and ends separately, and add them all together.

Ex. 1. To find the surface of a cube, the length of each side being 20 feet. Ans. 2400 feet.

Ex. 2. To find the whole surface of a triangular prism, whose length is 20 feet, and each side of its end or base 18 inches. Ans. 91·948 feet.

Ex. 3. To find the convex surface of a round prism, or cylinder, whose length is 20 feet, and diameter of its base is 2 feet. Ans. 125·664.

Ex. 4. What must be paid for lining a rectangular cistern with lead, at 2d. a pound weight, the thickness of the lead being such as to weigh 7lb. for each square foot of surface; the inside dimensions of the cistern being as follow, viz. the length 3 feet 2 inches, the breadth 2 feet 8 inches, and depth 2 feet 6 inches. Ans. 2l. 3s. $10\frac{1}{2}$d.

PROBLEM II.

To find the Surface of a Pyramid or Cone.

MULTIPLY the perimeter of the base by the slant height, or length of the side, and half the product will evidently be the surface of the sides, or the sum of the areas of all the triangles which form it. To which add the area of the end or base, if requisite.

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* The truth of this will easily appear, by considering that the sides of any prism are parallelograms, whose common length is the same as the length of the solid, and their breadths taken all together make up the perimeter of the ends of the same.

And the rule is evidently the same for the surface of a cylinder.

Ex. 1. What is the upright ſurface of a triangular pyramid, the ſlant height being 20 feet, and each ſide of the baſe 3 feet? Anſ. 90 feet.

Ex. 2. Required the convex ſurface of a cone, or circular pyramid, the ſlant height being 50 feet, and the diameter of its baſe $8\frac{1}{2}$ feet. Anſ. 667·59.

PROBLEM III.

To find the Surface of the Fruſtum of a Pyramid or Cone; being the lower part, when the top is cut off by a plane parallel to the baſe.

Add together the perimeters of the two ends, and multiply their ſum by the ſlant height, taking half the product for the anſwer.—As is evident, becauſe the ſides of the ſolid are trapezoids, having the oppoſite ſides parallel.

Ex. 1. How many ſquare feet are in the ſurface of the fruſtum of a ſquare pyramid, whoſe ſlant height is 10 feet; alſo, each ſide of the baſe or greater end being 3 feet 4 inches, and each ſide of the leſs end 2 feet 2 inches? Anſ. 110 feet.

Ex. 2. To find the convex ſurface of the fruſtum of a cone, the ſlant height of the fruſtum being $12\frac{1}{2}$ feet, and the circumferences of the two ends 6 and 8·4 feet. Anſ. 90 feet.

PROBLEM IV.

To find the Solid Content of any Priſm or Cylinder.

* Find the area of the baſe, or end, whatever the figure of it may be; and multiply it by the length of the priſm or cylinder, for the ſolid content.

Note. For a cube, take the cube of its ſide; and for a parallelopipedon, multiply the length, breadth and depth all together, for the content.

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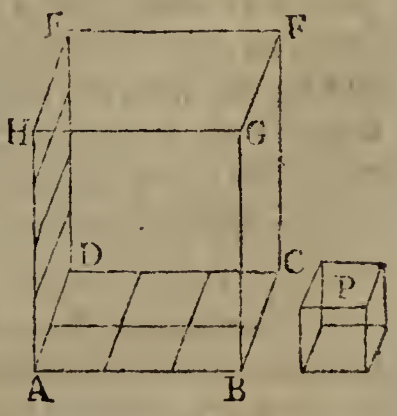
* This rule appears from the Geom. theor. 110, cor. 2, and is more particularly ſhewn as follows: Let the annexed rectangular parallelopipedon be the ſolid to be meaſured, and the cube P the ſolid meaſuring unit, its ſide being 1 inch, or 1 foot, &c.; alſo, let the length and breadth of the baſe ABCD, and the height AH, be divided into ſpaces equal to the length of the baſe of the cube P, namely here 3 in the length and 2 in the breadth, making 3 times 2 or



Ex. 1. To find the folid content of a cube, whofe fide is 24 inches. Anf. 13824.

Ex. 2. How many cubic feet are in a block of marble, its length being 3 feet 2 inches, breadth 2 feet 8 inches, and thicknefs 2 feet 6 inches? Anf. $21\frac{1}{9}$.

Ex. 3. How many gallons of water will the ciftern contain, whofe dimenfions are the fame as in the laft example, when 282 cubic inches are contained in one gallon? Anf. $129\frac{7}{47}$.

Ex. 4. Required the folidity of a triangular prifm, whofe length is 10 feet, and the three fides of its triangular end or bafe, are 3, 4, 5 feet. Anf. 60.

Ex. 5. Required the content of a round pillar, or cylinder, whofe length is 20 feet, and circumference 5 feet 6 inches. Anf. 48·1459.

PROBLEM V.

To find the Content of any Pyramid or Cone.

* FIND the area of the bafe, and multiply that area by the perpendicular height; then take $\frac{1}{3}$ of the product for the content.

Ex. 1. Required the folidity of the fquare pyramid, each fide of its bafe being 30, and its flant height 25. Anf. 6000.

Ex. 2. To find the content of a triangular pyramid, whofe perpendicular height is 30, and each fide of the bafe 3. Anf. 38·97117.

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or 6 fquares in the bafe AC, each equal to the bafe of the cube P. Hence it is manifeft that the parallelopipedon will contain the cube P, as many times as the bafe AC contains the bafe of the cube, repeated as often as the height AH contains the height of the cube. That is, the content of any parallelopipedon is found, by multiplying the area of the bafe by the altitude of that folid. And, becaufe all prifms and cylinders are equal to parallelopipedons of equal bafes and altitudes, by Geom. theor. 108, it follows that the rule is general for all fuch folids, whatever the figure of the bafe may be.

* This rule follows from that of the prifm, becaufe any pyramid is $\frac{1}{3}$ of a prifm of equal bafe and altitude; by Geom. theor. 115, cor. 1 and 2.

Ex. 3.

Ex. 3. To find the content of a triangular pyramid, its height being 14 feet 6 inches, and the three fides of its bafe 5, 6, 7 feet. Anf. 71·0352.

Ex. 4. What is the content of a pentagonal pyramid, its height being 12 feet, and each fide of its bafe 2 feet?
Anf. 27·5276.

Ex. 5. What is the content of the hexagonal pyramid, whofe height is 6·4 feet, and each fide of its bafe 6 inches?
Anf. 1·38564 feet.

Ex. 6. Required the content of a cone, its height being $10\frac{1}{2}$ feet, and the circumference of its bafe 9 feet.
Anf. 22·56093.

PROBLEM VI.

To find the Solidity of the Fruftum of a Cone or Pyramid.

* ADD into one fum, the areas of the two ends, and the mean proportional between them, or the fquare root of their product; and $\frac{1}{3}$ of that fum will be a mean area; which being multiplied by the perpendicular height or length of the fruftum, will give its content.

Ex. 1. To find the number of folid feet in a piece of timber, whofe bafes are fquares, each fide of the greater end

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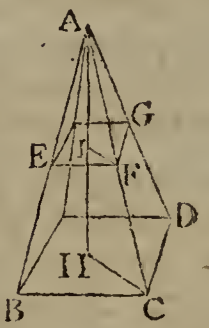
* Let ABCD be any pyramid, of which BCDGFE is a fruftum. And put a^2 for the area of the bafe BCD, b^2 the area of the top EFG, h the height IH of the fruftum, and c the height AI of the top part above it. Then $c + h =$ AH is the height of the whole pyramid.



Hence, by the laft prob. $\frac{1}{3}a^2(c+h)$ is the content of the whole pyramid ABCD, and $\frac{1}{3}b^2c$ the content of the top part AEFG; therefore the difference $\frac{1}{3}a^2(c+h) - \frac{1}{3}b^2c$ is the content of the fruftum BCDGFE. But, by Geom. theor. 112, $a^2 : b^2 :: (c+h)^2 : c^2$, or $a : b :: c+h : c$, and $a-b : b :: h : c$; hence $c = \frac{bh}{a-b}$, and $c + h = \frac{ah}{a-b}$; then thefe values of c and $c+h$ being fubftituted for them in the expreffion for the content of the fruftum, gives that content $= \frac{1}{3}a^2 \times \frac{ah}{a-b} - \frac{1}{3}b^2 \times \frac{bh}{a-b} = \frac{1}{3}h \times \frac{a^3 - b^3}{a-b} = \frac{1}{3}h \times (a^2 + ab + b^2)$; which is the rule above given.

being

being 15 inches, and each ſide of the leſs end 6 inches; alſo, the length or perpendicular altitude 24 feet? Anſ. $19\frac{1}{2}$.

Ex. 2. Required the content of a pentagonal fruſtum, whoſe height is 5 feet, each ſide of the baſe 18 inches, and each ſide of the top or leſs end 6 inches. Anſ. 9·31925 feet.

Ex. 3. To find the content of a conic fruſtum, the altitude being 18, the greateſt diameter 8, and the leaſt diameter 4. Anſ. 527·7888.

Ex. 4. What is the ſolidity of the fruſtum of a cone, the altitude being 25, alſo the circumference at the greater end being 20, and at the leſs end 10? Anſ. 464·205.

Ex. 5. If a caſk, which is two equal conic fruſtums joined together at the baſes, have its bung diameter 28 inches, the head diameter 20 inches, and length 40 inches; how many gallons of wine will it hold? Anſ. 79·0613.

PROBLEM VII.

To find the Surface of a Sphere, or any Segment.

* Rule I. Multiply the circumference of the ſphere by its diameter, and the product will be the whole ſurface of it.

Rule II.

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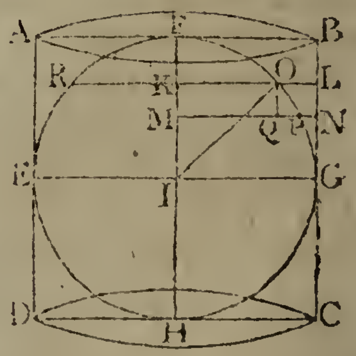
* Theſe rules come from the following theorems for the ſurface of a ſphere, viz. That the ſaid ſurface is equal to the curve ſurface of its circumſcribing cylinder; or that it is equal to 4 great circles of the ſame ſphere, or of the ſame diameter; which are thus proved.



Let ABCD be a cylinder, circumſcribing the ſphere EFGH, the former generated by the rotation of the rectangle FBCH about the axis or diameter FH; and the latter by the rotation of the ſemicircle FGH about the ſame diameter FH. Draw two lines KL, MN perpendicular to the axis, intercepting the parts LN, OP of the cylinder and ſphere; then ſhall the ring or cylindric ſurface generated by the rotation of LN, be equal to the ring or ſpherical ſurface generated by the arc OP. For, firſt, ſuppoſe the parallels KL and MN to be indefinitely near together; drawing IO, and alſo OQ parallel to LN. Then, the two triangles IKO, OQP being equiangular, it is,

RULE II. Multiply the ſquare of the diameter by 3·1416, and the product will be the ſurface.

Note. For the ſurface of a ſegment or fruſtum, multiply the whole circumference of the ſphere by the height of the part required.

Ex. 1. Required the convex ſuperficies of a ſphere, whoſe diameter is 7, and circumference 22. Anſ. 154.

Ex. 2. Required the ſuperficies of a globe, whoſe diameter is 24 inches. Anſ. 1809·5616.

Ex. 3. Required the area of the whole ſurface of the earth, its diameter being $7957\frac{3}{4}$ miles, and its circumference 25000 miles. Anſ. 198943750 ſq. miles.

Ex. 4. The axis of a ſphere being 42 inches, what is the convex ſuperficies of the ſegment whoſe height is 9 inches? Anſ. 1187·5248 inches.

Ex. 5. Required the convex ſurface of a ſpherical zone, whoſe breadth or height is 2 feet, and cut from a ſphere of $12\frac{1}{2}$ feet diameter. Anſ. 78·54 feet.

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is, as OP : OQ or LN :: IO or KL : KO :: circumference deſcribed by KL : circumf. deſcribed by KO; therefore the rectangle OP × circumf. of KO is equal to the rectangle LN × circumf. of KL; that is, the ring deſcribed by OP on the ſphere, is equal to the ring deſcribed by LN on the cylinder.

And as this is every where the caſe, therefore the ſums of any correſponding number of theſe are alſo equal; that is, the whole ſurface of the ſphere, generated by the whole ſemicircle FGH, is equal to the whole curve ſurface of the cylinder, generated by the height BC; as well as the ſurface of any ſegment generated by FO, equal to the ſurface of the correſponding ſegment generated by BL.

Corol. 1. Hence the ſurface of the ſphere is equal to 4 of its great circles, or equal to the circumference EFGH, or of DC, multiplied by the height BC, or the diameter FH.

Corol. 2. Hence alſo, the ſurface of any ſuch part as a ſegment, or fruſtum, or zone, is equal to the ſame circumference of the ſphere, multiplied by the height of the ſaid part. And conſequently ſuch ſpherical curve ſurfaces are to one another, in the ſame proportion as their altitudes.

PRO-

PROBLEM VIII.

To find the Solidity of a Sphere or Globe.

* Rule I. Multiply the surface by the diameter, and take $\frac{1}{6}$ of the product for the content.

Rule II. Take the cube of the diameter; and multiply it by the decimal ·5236, for the content.

Ex. 1. To find the content of a sphere whose axis is 12.
Ans. 904·7808.

Ex. 2. To find the solid content of the globe of the earth, supposing its circumference to be 25000 miles.
Ans. 263858149120 miles.

PROBLEM IX.

To find the Solid Content of a Spherical Segment.

† Rule I. From 3 times the diameter of the sphere, take double the height of the segment; then multiply the remainder

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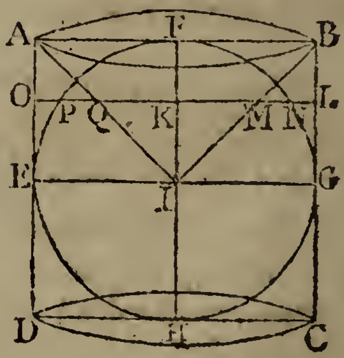
* For, put $d =$ the diameter, $c =$ the circuference, and $s =$ the surface of the sphere, or of its circumscribing cylinder; also, $a =$ the number 3·1416.

Then, $\frac{1}{4}s$ is $=$ the base of the cylinder, or one great circle of the sphere; and d is the height of the cylinder; therefore $\frac{1}{4}ds$ is the content of the cylinder. But $\frac{2}{3}$ of the cylinder is the sphere, by th. 117, Geom. that is, $\frac{2}{3}$ of $\frac{1}{4}ds$, or $\frac{1}{6}ds$ is the sphere; which is the first rule.

Again, because the surface s is $= ad^2$; therefore $\frac{1}{6}ds = \frac{1}{6}ad^3 =$ ·$5236d^3$, the content, as in the 2d rule.

† By corol. 3, of theor. 117, Geom. it appears that the spheric segment PFN, is equal to the difference between the cylinder ABLO, and the conic frustum ABMQ.



But, putting $d =$ AB or FH the diameter of the sphere or cylinder, $h =$ FK the height of the segment, $r =$ PK the radius of its base; and $a = 3·1416$; then the content of the cone ABI is $= \frac{1}{4}ad^2 \times \frac{1}{3}\text{FI} = \frac{1}{24}ad^3$; and by the similar cones ABI, QMI, as

$\text{FI}^3 : \text{KI}^3 :: \frac{1}{24}ad^3 : \frac{1}{24}ad^3 \times \left(\frac{\frac{1}{2}d - h}{\frac{1}{2}d}\right)^3 =$ the cone QMI;

therefore

remainder by the ſquare of the height, and the product by the decimal ·5236, for the content.

RULE II. To three times the ſquare of the radius of the ſegment's baſe, add the ſquare of its height; then multiply the ſum by the height, and the product by ·5236, for the content.

Ex. 1. To find the content of a ſpherical ſegment, of 2 feet in height, cut from a ſphere of 8 feet diameter.

Anſ. 41·888.

Ex. 2. What is the ſolidity of the ſegment of a ſphere, its height being 9, and the diameter of its baſe 20?

Anſ. 1795·4244.

Note. The general rules for meaſuring all ſorts of figures having been now delivered, we may next proceed to apply them to the ſeveral practical uſes in life, as follows.

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therefore the cone ABI — the cone QMI $= \frac{1}{24}ad^3 - \frac{1}{24}ad^3 \times \left(\frac{\frac{1}{2}d - h}{\frac{1}{2}d}\right)^3 = \frac{1}{4}ad^2h - \frac{1}{2}adh^2 + \frac{1}{3}ah^3$ is $=$ the conic fruſtum ABMQ.

And $\frac{1}{4}ad^2h$ is $=$ the cylinder ABLO.

Then the difference of theſe two is $\frac{1}{2}adh^2 - \frac{1}{3}ah^3 = \frac{1}{6}ah^2 \times (3d - 2h)$, for the ſpheric ſegment PFN; which is the firſt rule.

Again, becauſe $PK^2 = FK \times KH$, or $r^2 = h(d - h)$, therefore $d = \frac{r^2}{h} + h$, and $3d - 2h = \frac{3r^2}{h} + h = \frac{3r^2 + h^2}{h}$; which being ſubſtituted in the former rule, it becomes $\frac{1}{6}ah^2 \times \frac{3r^2 + h^2}{h} = \frac{1}{6}ah \times (3r^2 + h^2)$, which is the 2d rule.

Note. By ſubtracting a ſegment from a half ſphere, or from another ſegment, the content of any fruſtum or zone may be found.

LAND SURVEYING.

SECTION I.

DESCRIPTION AND USE OF THE INSTRUMENTS.

I. OF THE CHAIN.

LAND is meaſured with a chain, called Gunter's Chain, from its inventor, of 4 poles or 22 yards, or 66 feet in length. It conſiſts of 100 equal links; and the length of each link is therefore $\frac{22}{100}$ of a yard, or $\frac{66}{100}$ of a foot, or 7·92 inches.

Land is eſtimated in acres, roods, and perches. An acre is equal to 10 ſquare chains, that is, 10 chains in length and 1 chain in breadth. Or it is $220 \times 22 = 4840$ ſquare yards. Or it is $40 \times 4 = 160$ ſquare poles. Or it is $1000 \times 100 = 100000$ ſquare links. Theſe being all the ſame quantity.

Alſo, an acre is divided into 4 parts called roods, and a rood into 40 parts called perches, which are ſquare poles, or the ſquare of a pole of $5\frac{1}{2}$ yards long, or the ſquare of $\frac{1}{4}$ of a chain, or of 25 links, which is 625 ſquare links. So that the diviſions of land meaſure will be thus:

625 ſq. links	=	1 pole or perch
40 perches	=	1 rood
4 roods	=	1 acre.

The length of lines, meaſured with a chain, are beſt ſet down in links as integers, every chain in length being 100 links; and not in chains and decimals. Therefore after the content is found, it will be in ſquare links; then cut off five of the figures on the right-hand for decimals, and the reſt will be acres. Theſe decimals are then multiplied by 4 for roods, and the decimals of theſe again by 40 for perches.

EXAM. Suppoſe the length of a rectangular piece of ground be 792 links, and its breadth 385; to find the area in acres, roods, and perches.

792	3·04920
385	4
3960	·19680
6336	40
2376	7·87200
3·04920	

Anſ. 3 acres, 0 roods, 7 perches.

2. OF THE PLAIN TABLE.

THIS inſtrument conſiſts of a plain rectangular board, of any convenient ſize: the centre of which, when uſed, is fixed by means of ſcrews to a three-legged ſtand, having a ball and ſocket, or other joint, at the top, by means of which, when the legs are fixed on the ground, the table is inclined in any direction.

To the table belong various parts, as follow.

1. A frame of wood, made to fit round its edges, and to be taken off, for the convenience of putting a ſheet of paper upon the table. The one ſide of this frame is uſually divided into equal parts, for drawing lines acroſs the table, parallel or perpendicular to the ſides; and the other ſide of the frame is divided into 360 degrees from a centre which is in the middle of the table; by means of which the table is to be uſed as a theodolite, &c.

2. A needle and compaſs, either ſcrewed into the ſide of the table, or fixed beneath its centre, to point out the directions, and to be a check upon the ſights.

3. An index, which is a braſs two-foot ſcale, with either a ſmall teleſcope, or open ſights erected perpendicularly upon the ends. Theſe ſights and one edge of the index are in the ſame plane, and that edge is called the fiducial edge of the index.

To uſe this inſtrument, take a ſheet of paper which will cover it, and wet it to make it expand; then ſpread it flat on the table, preſſing down the frame upon the edges, to ſtretch it and keep it fixed there; and when the paper is become dry, it will, by contracting again, ſtretch itſelf ſmooth and flat from any cramps and unevenneſs. On this paper is to be drawn the plan or form of the thing meaſured.

Thus, begin at any part of the ground the moſt proper,

and make a point on a convenient part of the paper or table, to repreſent that point of the ground ; then fix in that point one leg of the compaſſes, or a fine ſteel pin, and apply to it the fiducial edge of the index, moving it round till through the ſights you perceive ſome remarkable object, as the corner of a field, &c ; and from the ſtation point draw a line with the point of the compaſſes along the fiducial edge of the index ; then ſet another object or corner, and draw its line ; do the ſame by another, and ſo on, till as many objects are ſet as may be thought fit. Then meaſure from the ſtation, towards as many of the objects as may be neceſſary, and no more, taking the requiſite offsets to corners or crooks in the hedges, laying the meaſures down on their reſpective lines on the table. Then, at any convenient place, meaſured to, fix the table in the ſame poſition, and ſet the objects which appear from thence, &c, as before ; and thus continue till the work is finiſhed, meaſuring ſuch lines as are neceſſary, and determining as many as may be, by interſecting lines of direction drawn from different ſtations.

Of ſhifting the Paper on the Plain Table.

When one paper is full, and you have occaſion for more ; draw a line in any manner through the fartheſt point of the laſt ſtation line, to which the work can be conveniently laid down ; then take the ſheet off the table, and fix another on, drawing a line upon it, in a part the moſt convenient for the reſt of the work ; then fold or cut the old ſheet by the line drawn on it, applying the edge to the line on the new ſheet, and as they lie in that poſition, continue the laſt ſtation line on the new paper, placing on it the reſt of the meaſure, beginning at where the old ſheet left off. And ſo on from ſheet to ſheet.

When the work is done, and you would faſten all the ſheets together into one piece, or rough plan, the aforeſaid lines are to be accurately joined together, in the ſame manner as when the lines were transferred from the old ſheets to the new ones.

But it is to be noted, that if the ſaid joining lines, on the old and new ſheets, have not the ſame inclination to the ſide of the table, the needle will not point to the original degree when the table is rectified ; and if the needle be required to reſpect ſtill the ſame degree of the compaſs, the eaſieſt way of drawing the lines in the ſame poſition, is to draw them both parallel to the ſame ſides of the table, by means of the equal diviſions marked on the other two ſides.

3. OF

3. OF THE THEODOLITE.

THE theodolite is a brazen circular ring, divided into 360 degrees, and having an index with ſights, or a teleſcope, placed on the centre, about which the index is moveable; alſo a compaſs fixed to the centre, to point out courſes and check the ſights; the whole being fixed by the centre on a ſtand of a convenient height for uſe.

In uſing this inſtrument, an exact account, or field-book, of all meaſures and things neceſſary to be remarked in the plan, muſt be kept, from which to make out the plan on returning home from the ground.

Begin at ſuch part of the ground, and meaſure in ſuch directions, as you judge moſt convenient; taking angles or directions to objects, and meaſuring ſuch diſtances as appear neceſſary, under the ſame reſtrictions as in the uſe of the plain table. And it is ſafeſt to fix the theodolite in the original poſition at every ſtation by means of fore and back objects, and the compaſs, exactly as in uſing the plain table; regiſtering the number of degrees cut off by the index when directed to each object; and, at any ſtation, placing the index at the ſame degree as when the direction towards that ſtation was taken from the laſt preceding one, to fix the theodolite there in the original poſition.

The beſt method of laying down the aforeſaid lines of direction, is to deſcribe a pretty large circle; then quarter it, and lay on it the ſeveral numbers of degrees cut off by the index in each direction, and drawing lines from the centre to all theſe marked points in the circle. Then, by means of a parallel ruler, draw, from ſtation to ſtation, lines parallel to the aforeſaid lines drawn from the centre to the reſpective points in the circumference.

4. OF THE CROSS.

THE croſs conſiſts of two pair of ſights ſet at right angles to each other, upon a ſtaff having a ſharp point at the bottom to ſtick in the ground.

The croſs is very uſeful to meaſure ſmall and crooked pieces of ground. The method is to meaſure a baſe or chief line, uſually in the longeſt direction of the piece, from corner to corner; and while meaſuring it, finding the places where perpendiculars would fall on this line, from the ſeveral corners and bends in the boundary of the piece, with the croſs, by fixing it, by trials, on ſuch parts of the line, ſo that through one pair of the ſights both ends of the line may appear, and through the other pair you can perceive the

correfponding bends or corners; and then meafuring the lengths of the faid perpendiculars.

REMARKS.

Befides the fore-mentioned inftruments, which are moft commonly ufed, there are fome others; as the circumferentor, which refembles the theodolite in fhape and ufe; and the femicircle, for taking angles, &c.

The perambulator is ufed for meafuring roads, and other great diftances on level ground, and by the fides of rivers. It has a wheel of $8\frac{1}{4}$ feet, or half a pole in circumference, by the turning of which the machine goes forward; and the diftance meafured, is pointed out by an index, which is moved round by clock work.

Levels, with telefcopic or other fights, are ufed to find the level between place and place, or how much one place is higher or lower than another. And in meafuring any floping or oblique line, either afcending or defcending, a fmall pocket level is ufeful for fhowing how many links for each chain are to be deducted, to reduce the line to the true horizontal length.

An offset-ftaff is a very ufeful and neceffary inftrument for meafuring the offsets and other fhort diftances. It is 10 links in length, being divided and marked at each of the 10 links.

Ten fmall arrows, or rods of iron or wood, are ufed to mark the end of every chain length, in meafuring lines. And fometimes pickets, or ftaves with flags, are fet up as marks or objects of direction.

Various fcales are alfo ufed in protracting and meafuring on the plan or paper; fuch as plane fcales, line of chords, protractor, compaffes, reducing fcale, parallel and perpendicular rules, &c. Of plane fcales, there fhould be feveral fizes, as a chain in 1 inch, a chain in $\frac{3}{4}$ of an inch, a chain in $\frac{1}{2}$ an inch, &c. And of thefe, the beft for ufe are thofe that are laid on the very edges of the ivory fcale, to mark off diftances, without compaffes.

5. OF THE FIELD-BOOK.

In furveying with the plane table, a field-book is not ufed, as every thing is drawn on the table immediately when it is meafured. But in furveying with the theodolite, or any other inftrument, fome fort of a field-book muft be ufed, to write down in it a regifter or account of all that is done and occurs relative to the furvey in hand.

This book every one contrives and rules as he thinks fitteft for himfelf. The following is a fpecimen of a form which has been formerly ufed. It is ruled into 3 columns: the middle, or principal column, is for the ftations, angles, bearings,

bearings, diſtances meaſured, &c; and thoſe on the right and left are for the offsets on the right and left, which are ſet againſt their correſponding diſtances in the middle column; as alſo for ſuch remarks as may occur, and may be proper to note in drawing the plan, &c.

Here ⊙ 1 is the firſt ſtation, where the angle or bearing is 105° 25′. On the left, at 73 links in the diſtance or principal line, is an offset of 92; and at 610 an offset of 24 to a croſs hedge. On the right, at 0, or the beginning, an offset 25 to the corner of the field; at 248 Brown's boundary hedge commences; at 610 an offset 35; and at 954, the end of the firſt line, the 0 denotes its terminating in the hedge. And ſo on for the other ſtations.

A line is drawn under the work, at the end of every ſtation line, to prevent confuſion.

Form of this Field-Book.

Offsets and Remarks on the left.	Stations, Bearings, and Diſtances.	Offsets and Remarks on the right.
	⊙ 1	
	105° 25′	
	00	25 corner
92	73	
	248	Brown's hedge
croſs a hedge 24	610	35
	954	00
	⊙ 2	
	53° 10′	
	00	00
houſe corner 51	25	21
	120	29 a tree
34	734	40 a ſtyle
	⊙ 3	
	67° 20′	
	61	35
a brook 30	248	
	639	16 a ſpring
foot-path 16	810	
croſs hedge 18	973	20 a pond

But

But ſome ſkilful ſurveyors now make uſe of a different method for the field-book, namely, beginning at the bottom of the page and writing upwards; by which they ſketch a neat boundary on either hand, as they paſs along; an example of which will be given further on, in the method of ſurveying a large eſtate.

In ſmaller ſurveys and meaſurements, a good way of ſetting down the work, is, to draw by the eye, on a piece of paper, a figure reſembling that which is to be meaſured; and ſo writing the dimenſions, as they are found, againſt the correſponding parts of the figure. And this method may be practiſed to a conſiderable extent, even in the larger ſurveys.

SECTION II.

THE PRACTICE OF SURVEYING.

THIS part contains the ſeveral works proper to be done in the field, or the ways of meaſuring by all the inſtruments, and in all ſituations.

PROBLEM I.

To Meaſure a Line or Diſtance.

To meaſure a line on the ground with the chain: Having provided a chain, with 10 ſmall arrows, or rods, to ſtick one into the ground, as a mark, at the end of every chain; two perſons take hold of the chain, one at each end of it; and all the 10 arrows are taken by one of them, who goes foremoſt, and is called the leader; the other being called the follower, for diſtinction's ſake.

A picket, or ſtation ſtaff, being ſet up in the direction of the line to be meaſured, if there do not appear ſome marks naturally in that direction; they meaſure ſtraight towards it, the leader fixing down an arrow at the end of every chain, which the follower always takes up, till all the ten arrows are uſed. They are then all returned to the leader, to uſe over again. And thus the arrows are changed from the one to the other at every 10 chains length, till the whole line is finiſhed; then the number of changes of the arrows ſhews the number of tens, to which the follower adds the arrows he

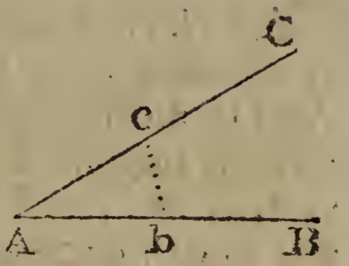
he holds in his hand, and the number of links of another chain over to the mark or end of the line. So, if there have been 3 changes of the arrows, and the follower hold 6 arrows, and the end of the line cut off 45 links more, the whole length of the line is ſet down in links thus, 3645.

When the ground is not level, but either aſcending or deſcending; at every chain length, lay the offset ſtaff, or link-ſtaff down in the ſlope of the chain, upon which lay the ſmall pocket level, to ſhew how many links or parts the ſlope line is longer than the true level one; then draw the chain forward ſo many links or parts, which reduces the line to the horizontal direction.

PROBLEM II.

To take Angles and Bearings.

LET B and C be two objects, or two pickets ſet up perpendicular, and let it be required to take their bearings, or the angle formed between them at any ſtation A.



1. *With the Plain Table.*

The table being covered with a paper, and fixed on its ſtand: plant it at the ſtation A, and fix a fine pin, or a point of the compaſſes, in a proper point of the paper, to repreſent the point A: Cloſe by the ſide of this pin lay the fiducial edge of the index, and turn it about, ſtill touching the pin, till one object B can be ſeen through the ſights: then by the fiducial edge of the index draw a line. In the very ſame manner draw another line in the direction of the other object C. And it is done.

2. *With the Theodolite, &c.*

Direct the fixed ſights along one of the lines, as AB, by turning the inſtrument about till the mark B is ſeen through theſe ſights; and there ſcrew the inſtrument faſt. Then turn the moveable index about till, through its ſights, you ſee the other mark C. Then the degrees cut by the index, upon the graduated limb or ring of the inſtrument, ſhew the quantity of the angle.

3. *With the Magnetic Needle and Compaſs.*

Turn the inſtrument, or compaſs, ſo, that the north end of the needle point to the flower-de-luce. Then direct the ſights

ſights to one mark as B, and note the degrees cut by the needle. Next direct the ſights to the other mark C, and note again the degrees cut by the needle. Then their ſum or difference, as the caſe is, will give the quantity of the angle BAC.

4. By Meaſurement with the Chain, &c.

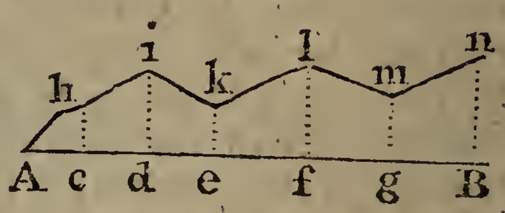
Meaſure one chain length, or any other length, along both directions, as to b and c. Then meaſure the diſtance b c, and it is done.—This is eaſily transferred to paper, by making a triangle A b c with theſe three lengths, and then meaſuring the angle A.

PROBLEM III.

To meaſure the Offsets.

A h i k l m n being a crooked hedge, or river, &c. From A meaſure in a ſtraight direction along the ſide of it to B. And in meaſuring along this line A B, obſerve when you are directly oppoſite any bends or corners of the boundary, as at c, d, e, &c; and from thence meaſure the perpendicular offsets c h, d i, &c, with the offset-ſtaff, if they are not very large, otherwiſe with the chain itſelf. And the work is done. The regiſter, or field-book, may be as follows:

Offs. left.		Baſe line AB.	
	0	⊙ A	
c h	62	45	A c
d i	84	220	A d
e k	70	340	A e
f l	98	510	A f
g m	57	634	A g
B n	91	785	A B

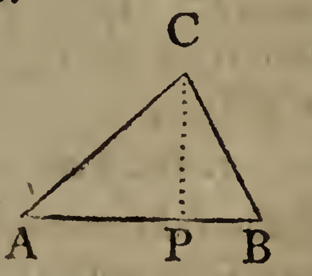


PROBLEM IV.

To ſurvey a Triangular Field ABC.

1. By the Chain.

AP 794
AB 1321
PC 826



Having

Having ſet up marks at the corners, which is to be done in all caſes where there are not marks naturally; meaſure with the chain from A to P, where a perpendicular would fall from the angle C, and ſet up a mark at P, noting down the diſtance AP. Then complete the diſtance AB by meaſuring from P to B. Having ſet down this meaſure, return to P, and meaſure the perpendicular PC. And thus, having the baſe and perpendicular, the area from them is eaſily found. Or having the place P of the perpendicular, the triangle is eaſily conſtructed.

Or, meaſure all the three ſides with the chain, and note them down. From which the content is eaſily found, or the figure conſtructed.

2. *By taking ſome of the Angles.*

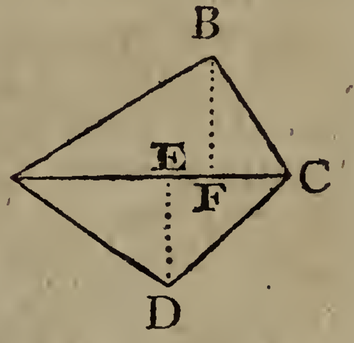
Meaſure two ſides AB, AC, and the angle A between them. Or meaſure one ſide AB, and the two adjacent angles A and B. From either of theſe ways the figure is eaſily planned; then by meaſuring the perpendicular CP on the plan, and multiplying it by half AB, you have the content.

PROBLEM V.

To meaſure a Four-ſided Field.

1. *By the Chain.*

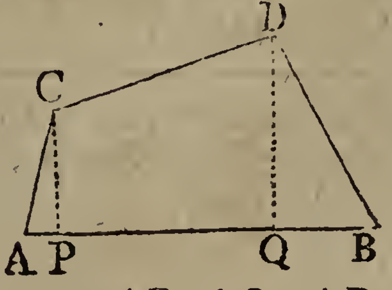
AE	214	210	DE
AF	362	306	BF
AC	592		



Meaſure along either of the diagonals, as AC; and either the two perpendiculars DE, BF, as in the laſt problem; or elſe the ſides AB, BC, CD, DA. From either of which the figure may be planned and computed as before directed.

Otherwiſe, by the Chain.

AP	110	352	PC
AQ	745	595	QD
AB	1110		



Meaſure on the longeſt ſide, the diſtances AP, AQ, AB; and the perpendicular PC, QD.

2. By

2. *By taking ſome of the Angles.*

Meaſure the diagonal AC (ſee the laſt fig. but one), and the angles CAB, CAD, ACB, ACD.—Or meaſure the four ſides, and any one of the angles, as BAD.

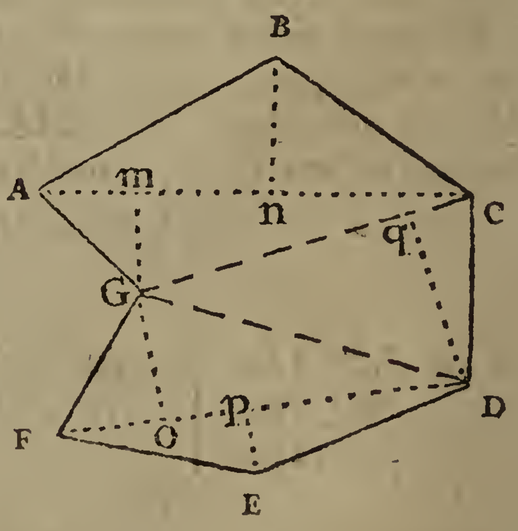
Thus			Or thus	
AC	591		AB	486
CAB	37°	20′	BC	394
CAD	41	15	CD	410
ACB	72	25	DA	462
ACD	54	40	BAD	78° 35′

PROBLEM VI.

To ſurvey any Field by the Chain only.

Having ſet up marks at the corners, where neceſſary, of the propoſed field ABCDEFG, walk over the ground, and conſider how it can beſt be divided in triangles and trapeziums; and meaſure them ſeparately as in the laſt two problems. Thus, the following figure is divided into the two trapeziums ABCG, GDEF, and the triangle GCD. Then, in the firſt trapezium, beginning at A, meaſure the diagonal AC, and the two perpendiculars G m, B n. Then the baſe GC, and the perpendicular D q. Laſtly, the diagonal DF, and the two perpendiculars p E, o G. All which meaſures write againſt the correſponding parts of a rough figure drawn to reſemble the figure to be ſurveyed, or ſet them down in any other form you chooſe.

Thus		Thus		
A m	135	130	m	G
A n	410	180	n	B
A c	550			
C q	152	230	q	D
C G	440			
F o	206	120	o	G
F p	288	80	p	E
F D	520			



Or

Or thus.

Meaſure all the ſides AB, BC, CD, DE, EF, FG, an GA; and the diagonals AC, CG, GD, DF.

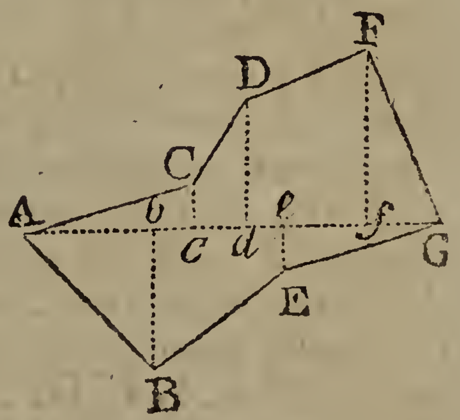
Otherwiſe.

Many pieces of land may be very well ſurveyed, by meaſuring any baſe line, either within or without them, together with the perpendiculars let fall upon it from every corner of them. For they are by thoſe means divided into ſeveral triangles and trapezoids, all whoſe parallel ſides are perpendicular to the baſe line; and the ſum of theſe triangles and trapeziums will be equal to the figure propoſed if the baſe line fall within it; if not, the ſum of the parts which are without being taken from the ſum of the whole which are both within and without, will leave the area of the figure propoſed.

In pieces that are not very large, it will be ſufficiently exact to find the points, in the baſe line, where the ſeveral perpendiculars will fall, by means of the *croſs*, and from thence meaſuring to the corners for the lengths of the perpendiculars.—And it will be moſt convenient to draw the line ſo as that all the perpendiculars may fall within the figure.

Thus, in the following figure, beginning at A, and meaſuring along the line AG, the diſtances and perpendiculars on the right and left, are as below.

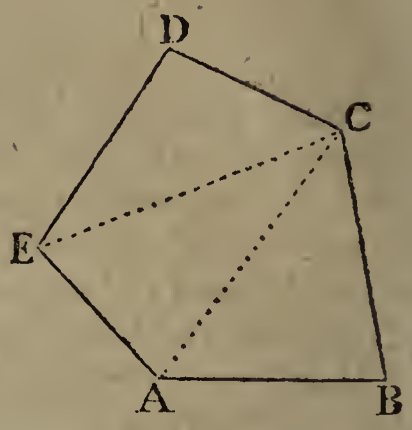
A b	315	350	b B
A c	440	70	c C
A d	585	320	d D
A e	610	50	e E
A f	990	470	f F
AG	1020	0	



PROBLEM VII.

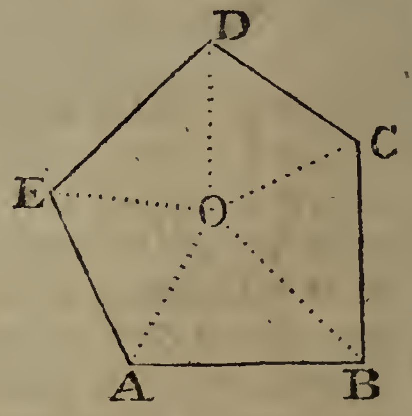
To furvey any Field with the Plain Table.

1. *From one Station.*



PLANT the table at any angle as C, from whence all the other angles, or marks fet up, can be feen; turn the table about till the needle point to the flower-de-luce; and there fcrew it faft. Make a point for C on the paper on the table, and lay the edge of the index to C, turning it about C till through the fights you fee the mark D; and by the edge of the index draw a dry or obfcure line: then meafure the diftance CD, and lay that diftance down on the line CD. Then turn the index about the point C, till the mark E be feen through the fights, by which draw a line, and meafure the diftance to E, laying it on the line from C to E. In like manner determine the pofitions of CA and CB, by turning the fights fucceffively to A and B; and lay the lengths of thofe lines down. Then connect the points with the boundaries of the field, by drawing the black lines CD, DE, EA, AB, BC.

2. *From a Station within the Field.*



When all the other parts cannot be feen from one angle, choofe fome place O within; or even without, if more convenient, from whence the other parts can be feen. Plant the table at O, then fix it with the needle north, and mark the point O on it. Apply the index fucceffively to O, turning it round with the fights to each angle A, B, C, D, E, drawing dry lines to them by the edge of the index; then meafuring the diftances OA, OB, &c, and laying them down upon thofe lines. Laftly, draw the boundaries AB, BC, CD, DE, EA.

3. *By going round the Figure.*

When the figure is a wood, or water, or from fome other obftruction you cannot meafure lines acrofs it; begin at any point

point A, and meaſure round it, either within or without the figure, and draw the directions of all the ſides thus: Plant the table at A, turn it with the needle to the north or flower-de-luce; fix it, and mark the point A. Apply the index to A, turning it till you can ſee the point E, and there draw a line; then the point B, and there draw a line: then meaſure theſe lines, and lay them down from A to E and B. Next, move the table to B, lay the index along the line AB, and turn the table about until you can ſee the mark A, and ſcrew faſt the table; in which poſition alſo the needle will again point to the flower-de-luce, as it will do indeed at every ſtation when the table is in the right poſition. Here turn the index about B till through the ſights you ſee the mark C; there draw a line, meaſure BC, and lay the diſtance upon that line after you have ſet down the table at C. Turn it then again into its proper poſition, and in like manner find the next line CD. And ſo on quite round by E to A again. Then the proof of the work will be the joining at A: for if the work is all right, the laſt direction EA on the ground, will paſs exactly through the point A on the paper; and the meaſured diſtance will alſo reach exactly to A. If theſe do not coincide, or nearly ſo, ſome error has been committed, and the work muſt be examined over again.

PROBLEM VIII.

To ſurvey a Field with the Theodolite, &c.

1. *From one Point or Station.*

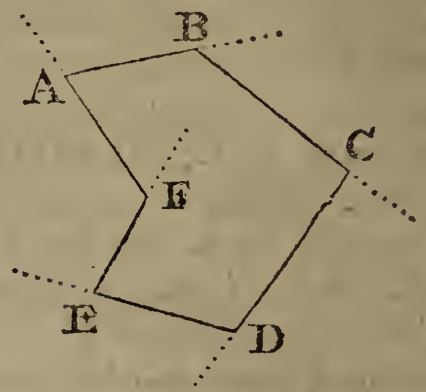
WHEN all the angles can be ſeen from one point, as the angle C (firſt fig. to laſt prob.) place the inſtrument at C, and turn it about till, through the fixed ſights, you ſee the mark B, and there fix it. Then turn the moveable index about till the mark A is ſeen through the ſights, and note the degrees cut on the inſtrument. Next turn the index ſucceſſively to E and D, noting the degrees cut off at each; which gives all the angles BCA, BCE, BCD. Laſtly, meaſure the lines CB, CA, CE, CD; and enter the meaſures in a field-book, or rather againſt the correſponding parts of a rough figure drawn by gueſs to reſemble the field.

2. *From a Point within or without.*

Plant the inſtrument at O (laſt fig.), and turn it about till the fixed ſights point to any object, as A; and there ſcrew it faſt. Then turn the moveable index round till the ſights

fights point fucceffively to the other points E, D, C, B, noting the degrees cut off at each of them; which gives all the angles round the point O. Laftly, meafure the diftances OA, OB, OC, OD, OE, noting them down as before, and the work is done.

3. *By going round the Field.*



By meafuring round, either within or without the field, proceed thus. Having fet up marks at B, C, &c, near the corners as ufual, plant the inftrument at any point A, and turn it till the fixed index be in the direction AB, and there fcrew it faft: then turn the moveable index to the direction AF; and the degrees cut off will be the angle A. Meafure the line AB, and plant the inftrument at B, and there in the fame manner obferve the angle A. Then meafure BC, and obferve the angle C. Then meafure the diftance CD, and take the angle D. Then meafure DE, and take the angle E. Then meafure EF, and take the angle F. And laftly meafure the diftance FA.

To prove the work; add all the inward angles A, B, C, &c, together, for when the work is right, their fum will be equal to twice as many right angles as the figure has fides, wanting 4 right angles. But when there is an angle, as F, that bends inwards, and you meafure the external angle, which is lefs than two right angles, fubtract it from four right angles, or 360 degrees, to give the internal angle greater than a femicircle or 180 degrees.

Otherwife.

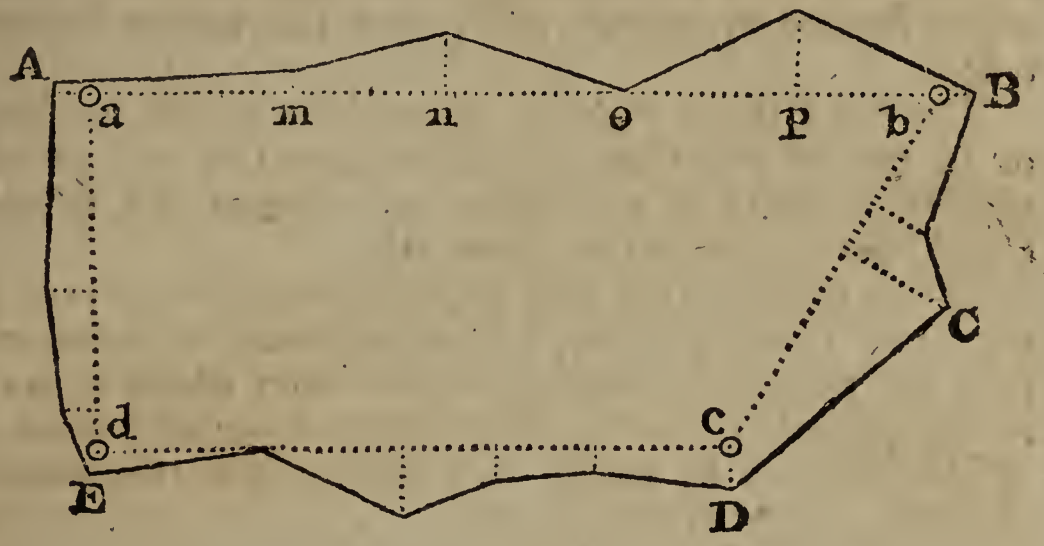
Inftead of obferving the internal angles, you may take the external angles, formed without the figure by producing the fides further out. And in this cafe, when the work is right, their fum altogether will be equal to 360 degrees. But when one of them, as F, runs inwards, fubtract it from the fum of the reft, to leave 360 degrees.

PROBLEM IX.

To furvey a Field with Crooked Hedges, &c.

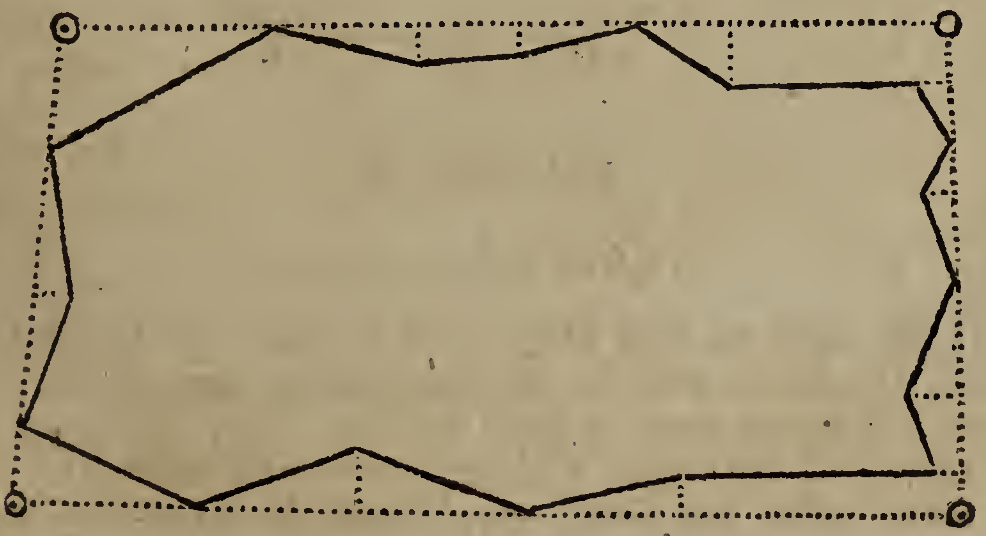
WITH any of the inftruments, meafure the lengths and pofitions of imaginary lines running as near the fides of the field as

as you can: and, in going along them, meaſure the offsets in the manner before taught; then you will have the plan on the paper in uſing the plain table, drawing the crooked hedges through the ends of the offsets; but in ſurveying with the theodolite, or other inſtrument, ſet down the meaſures properly in a field-book, or memorandum-book, and plan them after returning from the field, by laying down all the lines and angles.



So, in ſurveying the piece ABCDE, ſet up marks a, b, c, d, dividing it into as few ſides as may be. Then begin at any ſtation a, and meaſure the lines ab, bc, cd, da, taking their poſitions, or the angles a, b, c, d; and, in going along the lines, meaſure all the offsets, as at m, n, o, p, &c, along every ſtation line.

And this is done either within the field, or without, as may be moſt convenient. When there are obſtructions within, as wood, water, hills, &c, then meaſure without, as in the figure here below.



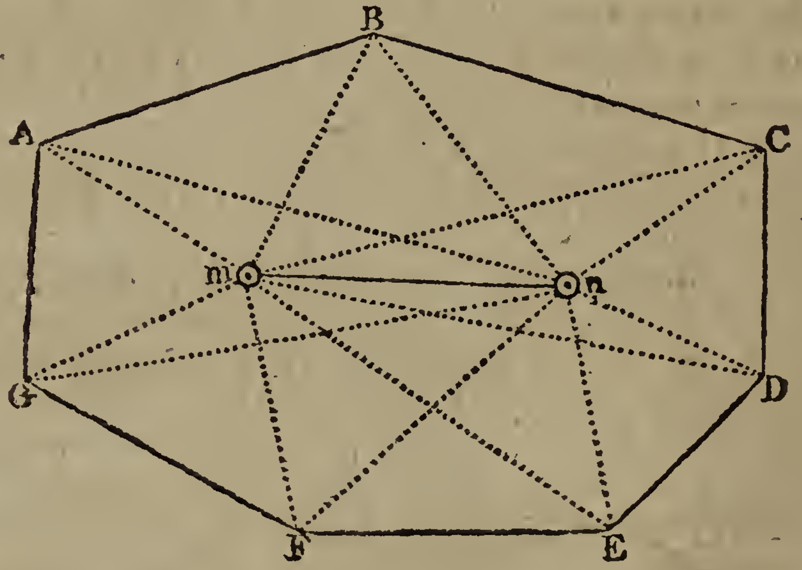
PROBLEM X.

To ſurvey a Field, or any other Thing, by two Stations.

THIS is performed by chooſing two ſtations, from whence all the marks and objects can be ſeen; then meaſuring the diſtance between the ſtations, and at each ſtation, taking the angles formed by every object, from the ſtation line or diſtance.

The two ſtations may be taken either within the bounds, or in one of the ſides, or in the direction of two of the objects, or quite at a diſtance and without the bounds of the objects, or part to be ſurveyed.

In this manner, not only grounds may be ſurveyed, without even entering them, but a map may be taken of the principal parts of a county, or the chief places of a town, or any part of a river or coaſt ſurveyed, or any other inacceſſible objects; by taking two ſtations, on two towers, or two hills, or ſuch like.



PROBLEM XI.

To ſurvey a Large Eſtate.

IF the eſtate be very large, and contain a great number of fields, it cannot well be done by ſurveying all the fields ſingly, and then putting them together; nor can it be done by taking all the angles and boundaries that incloſe it. For in theſe caſes, any ſmall errors will be ſo multiplied, as to render it very much diſtorted.

1. Walk

1. Walk over the eſtate two or three times, in order to get a perfect idea of it, and till you can carry the map of it tolerably well in your head. And to help your memory, draw an eye draught of it on paper, or at leaſt of the principal parts of it, to guide you; ſetting the names within the fields in that draught.

2. Chooſe two or more eminent places in the eſtate, for ſtations, from whence all the principal parts of it can be ſeen: and let theſe ſtations be as far diſtant from one another as poſſible.

3. Take ſuch angles, between the ſtations, as you think neceſſary, and meaſure the diſtances from ſtation to ſtation, always in a right line: theſe things muſt be done, till you get as many angles and lines as are ſufficient for determining all the points of ſtation. And in meaſuring any of theſe ſtation diſtances, mark accurately where theſe lines meet with any hedges, ditches, roads, lanes, paths, rivulets, &c; and where any remarkable object is placed, by meaſuring its diſtance from the ſtation line; and where a perpendicular from it cuts that line. And thus as you go along any main ſtation line, take offsets to the ends of all hedges, and to any pond, houſe, mill, bridge, &c, omitting nothing that is remarkable, and noting every thing down.

4. As to the inner parts of the eſtate, they muſt be determined in like manner, by new ſtation lines: for, after the main ſtations are determined, and every thing adjoining to them, then the eſtate muſt be ſubdivided into two or three parts by new ſtation lines; taking inner ſtations at proper places, where you can have the beſt view. Meaſure theſe ſtation lines as you did the firſt, and all their interſections with hedges, and offsets to ſuch objects as appear. Then proceed to ſurvey the adjoining fields, by taking the angles that the ſides make with the ſtation line, at the interſections, and meaſuring the diſtances to each corner, from the interſections. For the ſtation lines will be the baſes to all the future operations; the ſituation of all parts being entirely dependent upon them; and therefore they ſhould be taken of as great length as poſſible; and it is beſt for them to run along ſome of the hedges or boundaries of one or more fields, or to paſs through ſome of their angles. All things being determined for theſe ſtations, you muſt take more inner ſtations, and continue to divide and ſubdivide till at laſt you come to ſingle fields; repeating the ſame work for the inner ſtations, as for the outer ones, till all is done; and cloſe the work as often as you can, and in as few lines as poſſible.

5. An eſtate may be ſo ſituated that the whole cannot be ſurveyed together; becauſe one part of the eſtate cannot be ſeen from another. In this caſe, you may divide it into three or four parts, and ſurvey the parts ſeparately, as if they were lands belonging to different perſons; and at laſt join them together.

6. As it is neceſſary to protract or lay down the work as you proceed in it, you muſt have a ſcale of a due length to do it by. To get ſuch a ſcale, meaſure the whole length of the eſtate in chains; then conſider how many inches long the map is to be; and from theſe will be known how many chains you muſt have in an inch; then make the ſcale accordingly, or chooſe one already made.

The New Method of Surveying.

In the former method of meaſuring a large eſtate, the accuracy of it depends on the correctneſs of the inſtruments uſed in taking the angles. To avoid the errors incident to ſuch a multitude of angles, other methods have of late years been uſed by ſome few ſkilful ſurveyors: the moſt practical, expeditious, and correct, ſeems to be the following.

Chooſe two or more eminences, as grand ſtations, and meaſure a principal baſe line from one ſtation to the other, noting every hedge, brook, or other remarkable object as you paſs by it; meaſuring alſo ſuch ſhort perpendicular lines to the bends of hedges as may be near at hand. From the extremities of this baſe line, or from any convenient parts of the ſame, go off with other lines to ſome remarkable object ſituated towards the ſides of the eſtate, without regarding the angles they make with the baſe line or with one another; ſtill remembering to note every hedge, brook or other object that you paſs by. Theſe lines, when laid down by interſections, will with the baſe line form a grand triangle on the eſtate; ſeveral of which, if need be, being thus laid down, you may proceed to form other ſmaller triangles and trapezoids on the ſides of the former: and ſo on, until you finiſh with the encloſures individually.

This grand triangle being completed, and laid down on the rough plan paper, the parts, exterior as well as interior, are to be completed by ſmaller triangles and trapezoids.

In countries where the lands are encloſed with high hedges, and where many lanes paſs through an eſtate, a theodolite may be uſed to advantage, in meaſuring the angles of ſuch lands;

lands; by which means, a kind of ſkeleton of the eſtate may be obtained, and the lane-lines ſerve as the baſes of ſuch triangles and trapezoids as are neceſſary to fill up the interior parts.

The field-book is ruled into three columns. In the middle one are ſet down the diſtances on the chain line at which any mark, offset, or other obſervation is made; and in the right and left hand columns are entered the offsets and obſervations made on the right and left hand reſpectively of the chain line.

It is of great advantage, both for brevity and perſpicuity, to begin at the bottom of the leaf, and write upwards; denoting the croſſing of fences, by lines drawn acroſs the middle column, or only a part of ſuch a line on the right and left oppoſite the figures, to avoid confuſion; and the corners of fields, and other remarkable turns in the fences where offsets are taken to, by lines joining in the manner the fences do, as will be beſt ſeen by comparing the book with the plan annexed to the field-book following, p. 74.

The letter in the left-hand corner at the beginning of every line, is the mark or place meaſured *from;* and, that at the right hand corner at the end, is the mark meaſured *to:* But when it is not convenient to go exactly from a mark, the place meaſured from, is deſcribed *ſuch a diſtance* from *one mark* towards *another;* and where a mark is not meaſured to, the exact place is aſcertained by ſaying, turn to the right or left hand, *ſuch a diſtance* to *ſuch a mark*, it being always underſtood that thoſe diſtances are taken in the chain line.

The characters uſed, are ┌ for *turn to the right hand*, ┐ for *turn to the left hand*, and ^ placed over an offset, to ſhew that it is not taken at right angles with the chain line, but in the line with ſome ſtraight fence; being chiefly uſed when croſſing their directions, and it is a better way of obtaining their true places than by offsets at right angles.

When a line is meaſured whoſe poſition is determined, either by former work (as in the caſe of producing a given line, or meaſuring from one known place or mark to another) or by itſelf (as in the third ſide of a triangle) it is called a *faſt line*, and a double line acroſs the book is drawn at the concluſion of it; but if its poſition is not determined (as in the ſecond ſide of a triangle) it is called a *looſe line*, and a ſingle line is drawn acroſs the book. When a line becomes determined in poſition, and is afterwards continued, a double line half through the book is drawn.

When

When a loofe line is meafured, it becomes abfolutely neceffary to meafure fome line that will determine its pofition. Thus, the firft line *ah*, being the bafe of a triangle, is always determined; but the pofition of the fecond fide *hj*, does not become determined, till the third fide *jb* is meafured; then the triangle may be conftructed, and the pofition of both is determined.

At the beginning of a line, to fix a loofe line to the mark or place meafured from, the fign of turning to the right or left hand muft be added (as at *j* in the third line); otherwife a ftranger, when laying down the work, may as eafily conftruct the triangle *hjb* on the wrong fide of the line *ah*, as on the right one: but this error cannot be fallen into, if the fign above named be carefully obferved.

In choofing a line to fix a loofe one, care muft be taken that it does not make a very acute or obtufe angle; as in the triangle *p*B*r*, by the angle at B being very obtufe, a fmall deviation from truth, even the breadth of a point at *p* or *r*, would make the error at B, when conftructed, very confiderable; but by conftructing the triangle *p*B*q*, fuch a deviation is of no confequence.

Where the words *leave off* are written in the field-book, it is to fignify that the taking of offsets is from thence difcontinued; and of courfe fomething is wanting between that and the next offset.

The field-book for this method, and the plan drawn from it, are contained in the four following pages, engraven on copper-plates.

Field-

		1370	56 to e
		836	56
n		684	50
		1180	90 to g
		960	24
		930	n
		700	48
m		400	30
		1430	to i
		1290	40
		1004	36
		980	m
		610	24
k		280	32
		1794	to l
		1464	22
	19°	1050	
		920	32
		650	60
		350	48
a		0	14
		3074	to b
		2494	
		2100	l
	0	2072	
	54	1730	
	80	1530	
		1420	k
	56 + 30	1170	
	52	620	
j	32	280	40
		2574	j
		2494	
		2000	44
		1880	50
		1840	
	50	1794	i
	34 + 50	1464	
	76	1328	
	96	1240	
	52 + 34	1130	
	34	860	
h	66	190	
		4450	h
		3570	g
		2620	f
		2590	
		2210	
		2080	e
		1574	d
		1550	
		1510	c
		990	b
a		806	

		768	to A
		526	70
		496	
	70	460	
	40	124	
D		100	
		455	D
		400	76
C		48	10
		600	to r
		432	C
	56	160	
B	44	36	
B		152	to q
		480	B
p	24	160	
	0	1750	
		1600	44 to s
		1028	
		930	A
	44	666	
	70	310	z
d	60	236	
		2148	480 to b
		1950	y
		1836	
	128	1724	
	60	1600	
	30	1480	x
	0	1320	
	50	1110	
		1080	
		840	w
u		750	50
		4440	36
		4420	v
		3884	u
		3380	60
		2992	90
		2692	t
	120	2624	
		2592	
		2500	s leave off
		2070	56
		1900	
		1840	r
	60	1770	
		1320	q
		808	p
leave off	40	650	
	80	360	
o	20	170	
h produce by i		220	o
		190	46

		580	to v
	40	500	
	76	300	
F	76	100	
		390	to F
J	20	150	
		954	J
	15	850	
		730	to E
	30	490	
		340	60
	0	280	
I	20	170	50
		744	to H
		672	0
	70	450	0
a	50	15	
		1160	to v
	32	1000	
		890	
		780	32
		590	40
		570	I
		530	40
		376	H
		256	150 leave off
		190	64
G		144	130
		1676	G
		1676	30
		896	24
		632	
		620	50
180 from u towards v		588	F
		644	to f
D		488	32
		2280	
		2270	E
	20	2230	
	56	2050	
		2030	
		1970	130 to W
		1552	180
		1380	96
		950	110
		860	54

Plan from the foregoing Field Book.

PROBLEM XII.

To survey a County, or large Tract of Land.

1. CHOOSE two, three, or four eminent places, for ftations; fuch as the tops of high hills or mountains, towers, or church fteeples, which may be feen from one another; from which moft of the towns and other places of note may alfo be feen; and fo as to be as far diftant from one another as poffible. Upon thefe places raife beacons, or long poles, with flags of different colours flying at them, fo as to be vifible from all the other ftations.

2. At all the places, which you would fet down in the map, plant long poles with flags at them, of feveral colours, to diftinguifh the places from one another; fixing them on the tops of church fteeples, or the tops of houfes, or in the centres of leffer towns.

Thefe marks then being fet up at a convenient number of places, and fuch as may be feen from both ftations; go to one of thefe ftations, and, with an inftrument to take angles, ftanding at that ftation, take all the angles between the other ftation and each of thefe marks. Then go to the other ftation, and take all the angles between the firft ftation and each of the former marks, fetting them down with the others, each againft his fellow with the fame colour. You may, if you can, alfo take the angles at fome third ftation, which may ferve to prove the work, if the three lines interfect in that point where any mark ftands. The marks muft ftand till the obfervations are finifhed at both ftations; and then they muft be taken down, and fet up at frefh places. The fame operations muft be performed, at both ftations, for thefe frefh places; and the like for others. The inftrument for taking angles muft be an exceeding good one, made on purpofe with telefcopic fights; and of a good length of radius. A circumferentor is reckoned a good inftrument for this purpofe.

3. And though it be not abfolutely neceffary to meafure any diftance, becaufe a ftationary line being laid down from any fcale, all the other lines will be proportional to it; yet it is better to meafure fome of the lines, to afcertain the diftances of places in miles, and to know how many geometrical miles there are in any length; as alfo from thence to make a fcale to meafure any diftance in miles. In meafuring any diftance, it will not be exact enough to go along the high roads; by reafon of their turnings and windings, hardly

hardly ever lying in a right line between the ſtations, which muſt cauſe infinite reductions, and create endleſs trouble to make it a right line; for which reaſon it can never be exact. But a better way is to meaſure in a right line with a chain, between ſtation and ſtation, over hills and dales or level fields, and all obſtacles. Only in caſe of water, woods, towns, rocks, banks, &c, where one cannot paſs, ſuch parts of the line muſt be meaſured by the methods of inacceſſible diſtances; and beſides, allowing for aſcents and deſcents, when they are met with. A good compaſs that ſhews the bearing of the two ſtations, will always direct you to go ſtraight, when you do not ſee the two ſtations; and in the progreſs, if you can go ſtraight, offsets may be taken to any remarkable places, likewiſe noting the interſection of the ſtation line with all roads, rivers, &c.

4. From all the ſtations, and in the whole progreſs, be very particular in obſerving ſea-coaſts, river-mouths, towns, caſtles, houſes, churches, mills, trees, rocks, ſands, roads, bridges, fords, ferries, woods, hills, mountains, rills, brooks, parks, beacons, ſluices, floodgates, locks, &c, and in general every thing that is remarkable.

5. After you have done with the firſt and main ſtation lines, which command the whole county; you muſt then take inner ſtations, at ſome places already determined; which will divide the whole into ſeveral partitions: and from theſe ſtations you muſt determine the places of as many of the remaining towns as you can. And if any remain in that part, you muſt take more ſtations, at ſome places already determined; from which you may determine the reſt. And thus go through all the parts of the county, taking ſtation after ſtation, till we have determined all we want. And in general the ſtation diſtances muſt always paſs through ſuch remarkable points as have been determined before, by the former ſtations.

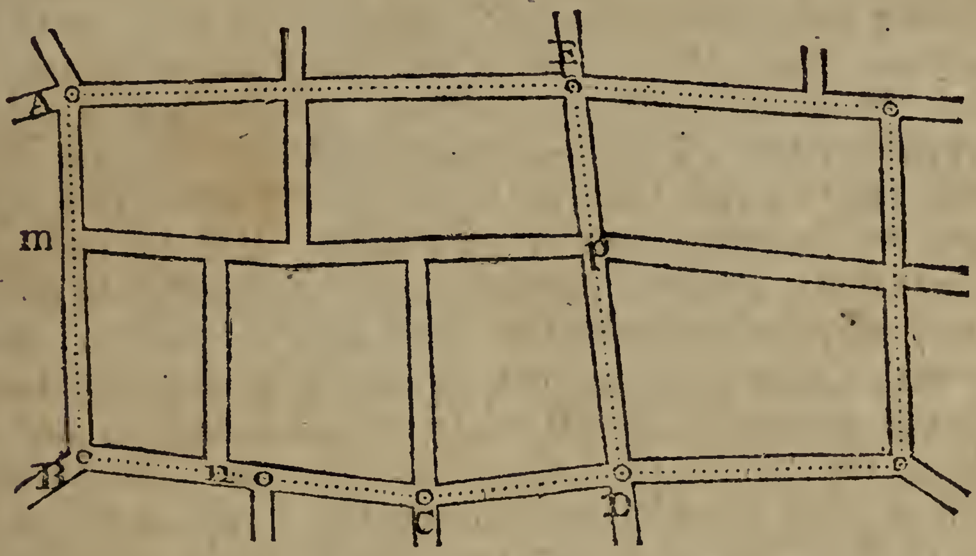
PROBLEM XIII.

To ſurvey a Town or City.

THIS may be done with any of the inſtruments for taking angles, but beſt of all with the plain table, where every minute part is drawn while in ſight. It is beſt alſo to have a chain of 50 feet long, divided into 50 links of one foot each, and an offset-ſtaff of 10 feet long.

Begin

Begin at the meeting of two or more of the principal ſtreets, through which you can have the longeſt proſpects, to get the longeſt ſtation lines: There having fixed the inſtrument, draw lines of direction along thoſe ſtreets, uſing two men as marks, or poles ſet in wooden pedeſtals, or perhaps ſome remarkable places in the houſes at the further ends, as windows, doors, corners, &c. Meaſure theſe lines with the chain, taking offsets with the ſtaff, at all corners of ſtreets, bendings, or windings, and to all remarkable things, as churches, markets, halls, colleges, eminent houſes, &c. Then remove the inſtrument to another ſtation, along one of theſe lines; and there repeat the ſame proceſs as before. And ſo on till the whole is finiſhed.



Thus, fix the inſtrument at A, and draw lines in the direction of all the ſtreets meeting there; then meaſure AB, noting the ſtreet on the left at m. At the ſecond ſtation B, draw the directions of the ſtreets meeting there; and meaſure from B to C, noting the places of the ſtreets at n and o as you paſs by them. At the 3d ſtation C, take the direction of all the ſtreets meeting there, and meaſure CD. At D do the ſame, and meaſure DE, noting the place of the croſs ſtreets at p. And in this manner go through all the principal ſtreets. This done, proceed to the ſmaller and intermediate ſtreets; and laſtly, to the lanes, alleys, courts, yards, and every part that it may be thought proper to repreſent in the plan.

SEC-

SECTION III.

Of PLANNING, COMPUTING, and DIVIDING.

PROBLEM I.

To lay down the Plan of any Survey.

If the ſurvey was taken with a plain table, you have a rough plan of it already on the paper which covered the table. But if the ſurvey was with any other inſtrument, a plan of it is to be drawn from the meaſures that were taken in the ſurvey, and firſt of all a rough plan on paper.

To do this, you muſt have a ſet of proper inſtruments, for laying down both lines and angles, &c ; as ſcales of various ſizes, the more of them, and the more accurate, the better ; ſcales of chords, protractors, perpendicular and parallel rulers, &c. Diagonal ſcales are beſt for the lines, becauſe they extend to three figures, or chains and links, which are hundredth parts of chains. But in uſing the diagonal ſcale, a pair of compaſſes muſt be employed to take off the lengths of the principal lines very accurately. But a ſcale with a thin edge divided, is much readier for laying down the perpendicular offsets to crooked hedges, and for marking the places of thoſe offsets upon the ſtation line ; which is done at only one application of the edge of the ſcale to that line, and then pricking off all at once the diſtances along it. Angles are to be laid down either with a good ſcale of chords, which is perhaps the moſt accurate way ; or with a large protractor, which is much readier when many angles are to be laid down at one point, as they are pricked off all at once round the edge of the protractor.

In general, all lines and angles muſt be laid down on the plan in the ſame order in which they were meaſured in the field, and in which they are written in the field-book ; laying down firſt the angles for the poſition of lines, next the lengths of the lines, with the places of the offsets, and then the lengths of the offsets themſelves, all with dry or obſcure lines ; then a black line drawn through the extremities of all the offsets, will be the hedge or bounding line of the field, &c. After the principal bounds and lines are laid down, and made to fit or cloſe properly, proceed next to the ſmaller objects, till you have entered every thing that ought to appear in the plan, as houſes, brooks, trees, hills, gates, ſtiles, roads, lanes, mills, bridges, woodlands, &c. &c.

The

The north side of a map or plan is commonly placed uppermoſt, and a meridian ſomewhere drawn, with the compaſs or flower-de-luce pointing north. Alſo, in a vacant part, a ſcale of equal parts or chains is drawn, with the title of the map in conſpicuous characters, and embelliſhed with a compartment. Hills are ſhadowed, to diſtinguiſh them in the map. Colour the hedges with different colours; repreſent hilly grounds by broken hills and valleys; draw ſingle dotted lines for foot-paths, and double ones for horſe or carriage roads. Write the name of each field and remarkable place within it, and, if you chooſe, its content in acres, roods, and perches.

In a very large eſtate, or a county, draw vertical and horizontal lines through the map, denoting the ſpaces between them by letters placed at the top, and bottom, and ſides, for readily finding any field or other object mentioned in a table.

In mapping counties, and eſtates that have uneven grounds of hills and valleys, reduce all oblique lines, meaſured up-hill and down-hill, to horizontal ſtraight lines, if that was not done during the ſurvey, before they were entered in the field-book, by making a proper allowance to ſhorten them. For which purpoſe there is commonly a ſmall table engraven on ſome of the inſtruments for ſurveying.

PROBLEM II.

To compute the Contents of Fields.

1. Compute the contents of the figures, whether triangles, or trapeziums, &c, by the proper rules for the ſeveral figures laid down in meaſuring; multiplying the lengths by the breadths, both in links, and divide by 2; the quotient is acres, after you have cut off five figures on the right for decimals. Then bring theſe decimals to roods and perches, by multiplying firſt by 4, and then by 40. An example of which is given in the deſcription of the chain, pag. 50.

2. In ſmall and ſeparate pieces, it is uſual to compute their contents from the meaſures of the lines taken in ſurveying them, without making a correct plan of them.

3. In pieces bounded by very crooked and winding hedges, meaſured by offsets, all the parts between the offsets are moſt accurately meaſured ſeparately as ſmall trapezoids.

4. Sometimes ſuch pieces as that laſt mentioned, are computed by finding a mean breadth, by dividing the ſum of the offsets

offsets by the number of them, accounting that for one of them where the boundary meets the ſtation line; then multiply the length by that mean breadth.

But this method is commonly in ſome degree erroneous.

5. But in larger pieces, and whole eſtates, conſiſting of many fields, it is the common practice to make a rough plan of the whole, and from it compute the contents quite independent of the meaſures of the lines and angles that were taken in ſurveying. For then new lines are drawn in the fields in the plan, ſo as to divide them into trapeziums and triangles, the baſes and perpendiculars of which are meaſured on the plan by means of the ſcale from which it was drawn, and ſo multiplied together for the contents. In this way, the work is very expeditiouſly done, and ſufficiently correct; for ſuch dimenſions are taken as afford the moſt eaſy method of calculation; and, among a number of parts, thus taken and applied to a ſcale, it is likely that ſome of the parts will be taken a ſmall matter too little, and others too great; ſo that they will, upon the whole, in all probability, very nearly balance one another. After all the fields, and particular parts, are thus computed ſeparately, and added all together into one ſum; calculate the whole eſtate independent of the fields, by dividing it into large and arbitrary triangles and trapeziums, and add theſe alſo together. Then if this ſum be equal to the former, or nearly ſo, the work is right; but if the ſums have any conſiderable difference, it is wrong, and they muſt be examined, and recomputed, till they nearly agree.

6. But the chief ſecret in computing, conſiſts in finding the contents of pieces bounded by curved or very irregular lines, or in reducing ſuch crooked ſides of fields or boundaries to ſtraight lines, that ſhall incloſe the ſame or equal area with thoſe crooked ſides, and ſo obtain the area of the curved figure by means of the right-lined one, which will commonly be a trapezium. Now this reducing the crooked ſides to ſtraight ones, is very eaſily and accurately performed in this manner:—Apply the ſtraight edge of a thin, clear piece of lanthorn-horn to the crooked line which is to be reduced, in ſuch a manner, that the ſmall parts cut off from the crooked figure by it, may be equal to thoſe which are taken in: which equality of the parts included and excluded you will preſently be able to judge of very nicely by a little practice: then with a pencil, or point of a tracer, draw a line by the ſtraight edge of the horn. Do the ſame by the other ſides of the field or figure. So ſhall you have a ſtraight

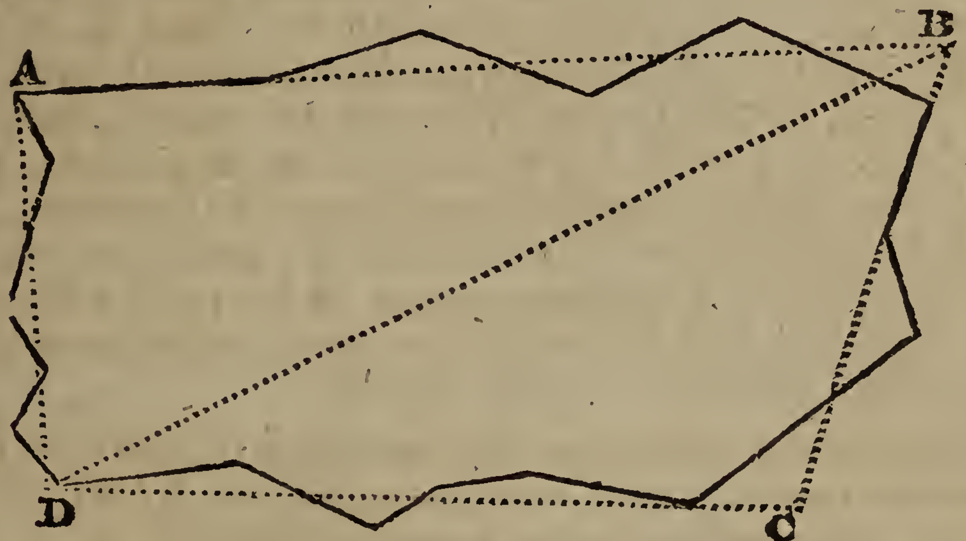
ſided

ſided figure equal to the curved one; the content of which, being computed as before directed, will be the content of the curved figure propoſed.

Or, inſtead of the ſtraight edge of the horn, a horſe-hair may be applied acroſs the crooked ſides in the ſame manner; and the eaſieſt way of uſing the hair, is to ſtring a ſmall ſlender bow with it, either of wire, or cane, or whale-bone, or ſuchlike ſlender elaſtic matter; for, the bow keeping it always ſtretched, it can be eaſily and neatly applied with one hand, while the other is at liberty to make two marks by the ſide of it, to draw the ſtraight line by.

EXAMPLE.

Thus, let it be required to find the contents of the ſame figure as in prob. IX of the laſt ſection, pag. 65, to a ſcale of 4 chains to an inch.



Draw the four dotted ſtraight lines AB, BC, CD, DA, cutting off equal quantities on both ſides of them, which they do as near as the eye can judge: ſo is the crooked figure reduced to an equivalent right-lined one of four ſides ABCD. Then draw the diagonal BD, which, by applying a proper ſcale to it, meaſures 1256. Alſo the perpendicular, or neareſt diſtance, from A to this diagonal, meaſures 456; and the diſtance of C from it, is 428.

Then, half the ſum of 456 and 428, multiplied by the diagonal 1256, gives 555152 ſquare links, or 5 acres, 2 roods, 8 perches, the content of the trapezium, or of the irregular crooked piece.

PRO-

PROBLEM III.

To transfer a Plan to another Paper, &c.

After the rough plan is completed, and a fair one is wanted; this may be done by any of the following methods.

First Method.—Lay the rough plan on the clean paper, keeping them always preſſed flat and cloſe together, by weights laid on them. Then, with the point of a fine pin or pricker, prick through all the corners of the plan to be copied. Take them aſunder, and connect the pricked points, on the clean paper, with lines; and it is done. This method is only to be practiſed in plans of ſuch figures as are ſmall and tolerably regular, or bounded by right lines.

Second Method.—Rub the back of the rough plan over with black lead powder; and lay the ſaid black part on the clean paper on which the plan is to be copied, and in the proper poſition. Then with the blunt point of ſome hard ſubſtance, as braſs, or ſuch like, trace over the lines of the whole plan; preſſing the tracer ſo much as that the black lead under the lines may be transferred to the clean paper: after which, take off the rough plan, and trace over the leaden marks with common ink, or with Indian ink—Or, inſtead of blacking the rough plan, you may keep conſtantly a blacked paper to lay between the plans.

Third Method.—Another method of copying plans, is by means of ſquares. This is performed by dividing both ends and ſides of the plan which is to be copied, into any convenient number of equal parts, and connecting the correſponding points of diviſion with lines; which will divide the plan into a number of ſmall ſquares. Then divide the paper, upon which the plan is to be copied, into the ſame number of ſquares, each equal to the former when the plan is to be copied of the ſame ſize, but greater or leſs than the others, in the proportion in which the plan is to be increaſed or diminiſhed, when of a different ſize. Laſtly, copy into the clean ſquares the parts contained in the correſponding ſquares of the old plan; and you will have the copy, either of the ſame ſize, or greater or leſs in any proportion.

Fourth Method.—A fourth method is by the inſtrument called a pentagraph, which alſo copies the plan in any ſize required.

Fifth Method.—But the neateſt method of any, at leaſt in copying a fair plan, is this. Procure a copying frame or glaſs, made in this manner; namely, a large ſquare of the beſt window glaſs, ſet in a broad frame of wood, which can be raiſed up to any angle, when the lower ſide of it reſts on a table. Set this frame up to any angle before you, facing a ſtrong light; fix the old plan and clean paper together with ſeveral pins quite around, to keep them together, the clean paper being laid uppermoſt, and over the face of the plan to be copied. Lay them, with the back of the old plan, on the glaſs, namely, that part which you intend to begin at to copy firſt; and, by means of the light ſhining through the papers, you will very diſtinctly perceive every line of the plan through the clean paper. In this ſtate then trace all the lines on the paper with a pencil. Having drawn that part which covers the glaſs, ſlide another part over the glaſs and copy it in the ſame manner. Then another part. And ſo on, till the whole is copied.

Then, take them aſunder, and trace all the pencil-lines over with a fine pen and Indian ink, or with common ink.

And thus you may copy the fineſt plan, without injuring it in the leaſt.

When the lines are copied on the clean paper, the next buſineſs is to write ſuch names, remarks, or explanations as may be judged neceſſary; laying down the ſcale for taking the lengths of any parts, a flower-de-luce to point out the direction, and the proper title ornamented with a compartment; illuſtrating or colouring every part in the manner that ſhall ſeem moſt natural, ſuch as ſhading rivers or brooks with crooked lines; drawing the repreſentations of trees, buſhes, hills, woods, hedges, houſes, gates, roads, &c, in their proper places; running a ſingle dotted line for a foot-path, and a double one for a carriage road; and either repreſenting the baſes or the elevations of buildings, &c.

Of ARTIFICERS' WORKS,

AND

TIMBER MEASURING.

I. Of the CARPENTER's or SLIDING RULE.

THE Carpenter's or Sliding Rule, is an inſtrument much uſed in meaſuring of timber and artificers' works, both for taking the dimenſions, and computing the contents.

The inſtrument conſiſts of two equal pieces, each a foot in length, which are connected together by a folding joint.

One ſide or face of the rule, is divided into inches, and eighths, or half quarters. On the ſame face alſo are ſeveral plane ſcales, divided into twelfth parts by diagonal lines; which are uſed in planning dimenſions that are taken in feet and inches. The edge of the rule is commonly divided decimally, or into tenths; namely, each foot into 10 equal parts, and each of theſe into 10 parts again: ſo that by means of this laſt ſcale, dimenſions are taken in feet, tenths and hundredths, and multiplied as common decimal numbers, which is the beſt way.

On the one part of the other face are four lines, marked A, B, C, D; the two middle ones B and C being on a ſlider, which runs in a groove made in the ſtock. The ſame numbers ſerve for both theſe two middle lines, the one being above the numbers, and the other below.

Theſe four lines are logarithmic ones, and the three A, B, C, which are all equal to one another, are double lines, as they proceed twice over from 1 to 10. The other or loweſt line D, is a ſingle one, proceeding from 4 to 40. It is alſo called the girt line, from its uſe in computing the contents of trees and timber; and upon it are marked WG at 17·15, and AG at 18·95, the wine and ale gage points, to make this inſtrument ſerve the purpoſe of a gaging rule.

On the other part of this face, there is a table of the value of a load, or 50 cubic feet of timber, at all prices, from 6 pence to 2 ſhillings a foot.

When

When 1 at the beginning of any line is accounted 1, then the 1 in the middle will be 10, and the 10 at the end 100; but when 1 at the beginning is accounted 10, then the 1 in the middle is 100, and the 10 at the end 1000; and so on. And all the smaller divisions are altered proportionally.

II. ARTIFICERS' WORK.

ARTIFICERS compute the contents of their works by several different measures. As,

Glazing and masonry by the foot;
Painting, plastering, paving, &c, by the yard, of 9 square feet;
Flooring, partitioning, roofing, tiling, &c, by the square, of 100 square feet.

And brickwork, either by the yard of 9 square feet, or by the perch, or square rod or pole, containing $272\frac{1}{4}$ square feet, or $30\frac{1}{4}$ square yards, being the square of the rod or pole of $16\frac{1}{2}$ feet or $5\frac{1}{2}$ yards long.

As this number $272\frac{1}{4}$ is troublesome to divide by, the $\frac{1}{4}$ is often omitted in practice, and the content in feet divided only by the 272. But when the exact divisor $272\frac{1}{4}$ is to be used, it will be easier to multiply the feet by 4, and then divide successively by 9, 11, and 11. Also to divide square yards by $30\frac{1}{4}$, first multiply them by 4, and then divide twice by 11.

All works, whether superficial or solid, are computed by the rules proper to the figure of them, whether it be a triangle, or rectangle, a parallelopiped, or any other figure.

III. BRICKLAYERS' WORK.

BRICKWORK is estimated at the rate of a brick and a half thick. So that, if a wall be more or less than this standard thickness, it must be reduced to it, as follows:

Multiply the superficial content of the wall by the number of half bricks in the thickness, and divide the product by 3.

 The

The dimenfions of a building are ufually taken by meafuring half round on the outfide and half round it on the infide; the fum of thefe two gives the compafs of the wall, to be multiplied by the height, for the content of the materials.

Chimnies are by fome meafured as if they were folid, deducting only the vacuity from the hearth to the mantle, on account of the trouble of them.

And by others they are girt or meafured round for their breadth, and the height of the ftory is their height, taking the depth of the jambs for their thicknefs. And in this cafe, no deduction is made for the vacuity from the floor to the mantle-tree, becaufe of the gathering of the breaft and wings, to make room for the hearth in the next ftory.

To meafure the chimney fhafts, which appear above the building; girt them about with a line for the breadth, to multiply by their height. And account their thicknefs half a brick more than it really is, in confideration of the plaftering and fcaffolding.

All windows, doors, &c, are to be deducted out of the contents of the walls in which they are placed. But this deduction is made only with regard to materials; for the whole meafure is taken for workmanfhip, and that all outfide meafure too, namely, meafuring quite round the outfide of the building, being in confideration of the trouble of the returns or angles. There are alfo fome other allowances, fuch as double meafure for feathered gable ends, &c.

EXAMPLES.

EXAM. I. How many yards and rods of ftandard brickwork are in a wall whofe length or compafs is 57 feet 3 inches, and height 24 feet 6 inches; the walls being $2\frac{1}{2}$ bricks or 5 half bricks thick? Anf. 8 rods, $17\frac{2}{3}$ yards.

EXAM. II. Required the content of a wall 62 feet 6 inches long, and 14 feet 8 inches high, and $2\frac{1}{2}$ bricks thick? Anf. 169.753 yards.

EXAM. III. A triangular gable is raifed $17\frac{1}{2}$ feet high, on an end wall whofe length is 24 feet 9 inches, the thicknefs being 2 bricks; required the reduced content? Anf. $32 \cdot 08\frac{1}{3}$ yards.

EXAM. IV. The end wall of a houfe is 28 feet 10 inches long, and 55 feet 8 inches high, to the eaves; 20 feet high is $2\frac{1}{2}$ bricks thick, other 20 feet high is 2 bricks thick, and the remaining 15 feet 8 inches is $1\frac{1}{2}$ brick thick; above which is a triangular gable, of 1 brick thick, which rifes 42 courfes of bricks, of which every 4 courfes make a foot. What is the whole content in ftandard meafure? Anf. 253·626 yards.

IV. MA-

IV. MASONS' WORK.

To maſonry belong all ſorts of ſtone-work; and the meaſure made uſe of is a foot, either ſuperficial or ſolid.

Walls, columns, blocks of ſtone or marble, &c, are meaſured by the cubic foot; and pavements, ſlabs, chimney-pieces, &c, by the ſuperficial or ſquare foot.

Cubic or ſolid meaſure is uſed for the materials, and ſquare meaſure for the workmanſhip.

In the ſolid meaſure, the true length, breadth, and thickneſs, are taken, and multiplied continually together. In the ſuperficial, there muſt be taken the length and breadth of every part of the projection, which is ſeen without the general upright face of the building.

EXAMPLES.

EXAM. I. Required the ſolid content of a wall, 53 feet 6 inches long, 12 feet 3 inches high, and 2 feet thick?

Anſ. $1310\frac{3}{4}$ feet.

EXAM. II. What is the ſolid content of a wall, the length being 24 feet 3 inches, height 10 feet 9 inches, and 2 feet thick? Anſ. 521·375 feet.

EXAM. III. Required the value of a marble ſlab, at 8s. per foot; the length being 5 feet 7 inches, and breadth 1 foot 10 inches? Anſ. 4l. 1s. $10\frac{1}{2}$d.

EXAM. IV. In a chimney-piece, ſuppoſe the

		feet	inches
length of the mantle and ſlab,	each	4	6
breadth of both together	-	3	2
length of each jamb	- -	4	4
breadth of both together	-	1	9

Required the ſuperficial content? Anſ. 21 feet 10 inches.

V. CARPENTERS' AND JOINERS' WORK.

To this branch belongs all the wood-work of a houſe, ſuch as flooring, partitioning, roofing, &c.

Large and plain articles are uſually meaſured by the ſquare foot or yard, &c; but enriched mouldings, and ſome other articles, are often eſtimated by running or lineal meaſure, and ſome things are rated by the piece.

 In

In meafuring of joifts, it is to be obferved, that only one of their dimenfions is the fame with that of the floor; for the other exceeds the length of the room by the thicknefs of the wall, and $\frac{1}{3}$ of the fame, becaufe each end is let into the wall about $\frac{2}{3}$ of its thicknefs.

No deductions are made for hearths, on account of the additional trouble and wafte of materials.

Partitions are meafured from wall to wall for one dimenfion, and from floor to floor, as far as they extend, for the other.

No deduction is made for door-ways, on account of the trouble of framing them.

In meafuring of joiners' work, the ftring is made to ply clofe to every part of the work over which it paffes.

The meafure of centering for cellars is found, by making a ftring pafs over the furface of the arch for the breadth, and taking the length of the cellar for the length; but in groin centering, it is ufual to allow double meafure, on account of their extraordinary trouble.

In roofing, the length of the houfe in the infide, together with $\frac{2}{3}$ of the thicknefs of one gable, is to be confidered as the length; and the breadth is equal to double the length of a ftring which is ftretched from the ridge down the rafter, and along the eaves-board, till it meets with the top of the wall.

For ftair-cafes, take the breadth of all the fteps, by making a line ply clofe over them, from the top to the bottom, and multiply the length of this line by the length of a ftep, for the whole area.—By the length of a ftep is meant the length of the front and the returns at the two ends; and by the breadth, is to be underftood the girt of its two outer furfaces, or the tread and rifer.

For the baluftrade, take the whole length of the upper part of the hand-rail, and girt over its end till it meet the top of the newel-poft, for the length; and twice the length of the balufter upon the landing, with the girt of the hand-rail, for the breadth.

For wainfcotting, take the compafs of the room for the length; and the height from the floor to the ceiling, making the ftring ply clofe into all the mouldings, for the breadth.—Out of this muft be made deductions for windows, doors, and chimneys, &c; but workmanfhip is counted for the whole, on account of the extraordinary trouble.

For doors, it is ufual to allow for their thicknefs, by adding it into both the dimenfions of length and breadth, and then multiply

multiply them together for the area.—If the door be panneled on both ſides, take double its meaſure for the workmanſhip; but if one ſide only be panneled, take the area and its half for the workmanſhip.—*For the ſurrounding architrave*, gird it about the outermoſt part for its length; and meaſure over it, as far as it can be ſeen when the door is open, for the breadth.

Window ſhutters, baſes, &c, are meaſured in the ſame manner.

In the meaſuring of roofing for workmanſhip alone, holes for chimey ſhafts and ſky-lights are generally deducted.

But in meaſuring for work and materials, they commonly meaſure in all ſky-lights, luthern-lights, and holes for the chimney ſhafts, on account of their trouble and waſte of materials.

EXAMPLES.

EXAM. I. Required the content of a floor 48 feet 6 inches long, and 24 feet 3 inches broad? Anſ. 11 ſq. $76\frac{1}{8}$ feet.

EXAM. II. A floor being 36 feet 3 inches long, and 16 feet 6 inches broad, how many ſquares are in it?

Anſ. 5 ſq. $98\frac{1}{8}$ feet.

EXAM. III. How many ſquares are there in 173 feet 10 inches in length, and 10 feet 7 inches height, of partitioning? Anſ. 18·3973 ſquares.

EXAM. IV. What coſt the roofing of a houſe at 10s. 6d. a-ſquare; the length, within the walls, being 52 feet 8 inches, and the breadth 30 feet 6 inches; reckoning the roof $\frac{3}{2}$ of the flat? Anſ. 12l. 12s. $11\frac{1}{4}$d.

EXAM. V. To how much at 6s. per ſquare yard, amounts the wainſcotting of a room; the height, taking in the cornice and mouldings, being 12 feet 6 inches, and the whole compaſs 83 feet 8 inches; alſo the three window ſhutters are each 7 feet 8 inches by 3 feet 6 inches, and the door 7 feet by 3 feet 6 inches; the door and ſhutters, being worked on both ſides, are reckoned work and half work?

Anſ. 36l. 12s. $2\frac{1}{4}$d.

VI. SLATERS' AND TILERS' WORK.

IN theſe articles, the content of a roof is found by multiplying the length of the ridge by the girt over from eaves to eaves; making allowance in this girt for the double row of ſlates at the bottom, or for how much one row of ſlates or tiles is laid over another.

When

When the roof is of a true pitch, that is, forming a right angle at top; then the breadth of the building with its half added, is the girt over both ſides.

In angles formed in a roof, running from the ridge to the eaves, when the angle bends inwards, it is called a valley; but when outwards, it is called a hip.

Deductions are made for chimney ſhafts or window holes.

EXAMPLES.

Exam. I. Required the content of a ſlated roof, the length being 45 feet 9 inches, and the whole girt 34 feet 3 inches?

Anſ. $174\frac{1}{10}$ yards.

Exam. II. To how much amounts the tiling of a houſe, at 25s. 6d. per ſquare; the length being 43 feet 10 inches, and the breadth on the flat 27 feet 5 inches, alſo the eaves projecting 16 inches on each ſide, and the roof of a true pitch?

Anſ. 24l. 9s. $5\frac{1}{4}$d.

VII. PLASTERERS' WORK.

Plasterers' work is of two kinds, namely, ceiling, which is plaſtering upon laths; and rendering, which is plaſtering upon walls: which are meaſured ſeparately.

The contents are eſtimated either by the foot or yard, or ſquare of 100 feet. Inriched mouldings, &c, are rated by running or lineal meaſure.

Deductions are to be made for chimneys, doors, windows, &c. But the windows are ſeldom deducted, as the plaſtered returns at the top and ſides are allowed to compenſate for the window opening.

EXAMPLES.

Exam. I. How many yards contains the ceiling, which is 43 feet 3 inches long, and 25 feet 6 inches broad?

Anſ. $122\frac{1}{2}$.

Exam. II. To how much amounts the ceiling of a room, at 10d. per yard; the length being 21 feet 8 inches, and the breadth 14 feet 10 inches?

Anſ. 1l. 9s. $8\frac{3}{4}$d.

Exam. III. The length of a room is 18 feet 6 inches, the breadth 12 feet 3 inches, and height 10 feet 6 inches; to how much amounts the ceiling and rendering, the former at 8d. and the latter at 3d. per yard; allowing for the door of 7 feet by 3 feet 8, and a fire-place of 5 feet ſquare?

Anſ. 1l. 13s. $3\frac{1}{4}$d.

Exam.

Exam. iv. Required the quantity of plaſtering in a room, the length being 14 feet 5 inches, breadth 13 feet 2 inches, and height 9 feet 3 inches to the under ſide of the cornice, which girts $8\frac{1}{2}$ inches, and projects 5 inches from the wall on the upper part next the ceiling; deducting only for a door 7 feet by 4?

Anſ.	53 yards	5 feet	3 inches	of rendering
	18	5	6	of ceiling
		39	$0\frac{1}{12}$	of cornice.

VIII. PAINTERS' WORK.

Painters' work is computed in ſquare yards. Every part is meaſured where the colour lies; and the meaſuring line is forced into all the mouldings, and corners.

Windows are done at ſo much a piece. And it is uſual to allow double meaſure for carved mouldings, &c.

EXAMPLES.

Exam. i. How many yards of painting contains the room which is 65 feet 6 inches in compaſs, and 12 feet 4 inches high? Anſ. $89\frac{3}{4}$ yards.

Exam. ii. The length of a room being 20 feet, its breadth 14 feet 6 inches, and height 10 feet 4 inches; how many yards of painting are in it, deducting a fire-place of 4 feet by 4 feet 4 inches, and two windows each 6 feet by 3 feet 2 inches? Anſ. $73\frac{2}{27}$ yards.

Exam. iii. What coſt the painting of a room, at 6d. per yard; its length being 24 feet 6 inches, its breadth 16 feet 3 inches, and height 12 feet 9 inches; alſo the door is 7 feet by 3 feet 6, and the window ſhutters to two windows each 7 feet 9 by 3 feet 6, but the breaks of the windows themſelves are 8 feet 6 inches high, and 1 foot 3 inches deep: deducting the fire-place of 5 feet by 5 feet 6? Anſ. 3l. 3s. $10\frac{3}{4}$d.

IX. GLAZIERS' WORK.

Glaziers take their dimenſions, either in feet, inches and parts, or feet, tenths, and hundredths. And they compute their work in ſquare feet.

In

In taking the length and breadth of a window, the crofs bars between the fquares are included. Alfo windows of round or oval forms are meafured as fquare, meafuring them to their greateft length and breadth, on account of the wafte in cutting the glafs.

EXAMPLES.

EXAM. I. How many fquare feet contains the window which is 4·25 feet long, and 2·75 feet broad? Anf. $11\frac{2}{3}$.

EXAM. II. What will the glazing a triangular fky-light come to, at 10d. per foot; the bafe being 12 feet 6 inches, and the perpendicular height 6 feet 9 inches?

Anf. 1l. 15s. $1d\frac{3}{4}$.

EXAM. III. There is a houfe with three tier of windows, three windows in each tier, their common breadth 3 feet 11 inches;

		feet	inches
now the height of the firft tier is		7	10
	of the fecond	6	8
	of the third	5	4

Required the expence of glazing at 14d. per foot?

Anf. 13l. 11s. $10\frac{1}{2}$d.

EXAM. IV. Required the expence of glazing the windows of a houfe at 13d. a foot; there being three ftories, and three windows in each ftory;

		feet	inches
the height of the lower tier is		7	9
	of the middle	6	6
	of the upper	5	$3\frac{1}{4}$
and of an oval window over the door		1	$10\frac{1}{2}$

the common breadth of all the windows being 3 feet 9 inches? Anf. 12l. 5s. 6d.

X. PAVERS' WORK.

PAVERS' work is done by the fquare yard. And the content is found by multiplying the length by the breadth.

EXAMPLES.

EXAM. I. What coft the paving a foot-path at 3s. 4d. a-yard; the length being 35 feet 4 inches, and breadth 8 feet 3 inches? Anf. 5l. 7s. $11\frac{1}{2}$d.

EXAM. II. What coft the paving a court, at 3s. 2d. per yard; the length being 27 feet 10 inches, and the breadth 14 feet 9 inches? Anf. 7l. 4s. $5\frac{1}{4}$d.

EXAM. III.

EXAM. III. What will be the expence of paving a rectangular court-yard, whofe length is 63 feet, and breadth 45 feet; in which there is laid a foot-path of 5 feet 3 inches broad, running the whole length, with broad ftones, at 3s. a-yard; the reft being paved with pebbles at 2s. 6d. a-yard?

Anf. 40l. 5s. $10\frac{1}{2}$d.

XI. PLUMBERS' WORK.

PLUMBERS' work is rated at fo much a pound, or elfe by the hundred weight, of 112 pounds.

Sheet lead ufed in roofing, guttering, &c, is from 7 to 10lb. to the fquare foot. And a pipe of an inch bore, is commonly 13 or 14lb. to the yard in length.

EXAMPLES.

EXAM. I. How much weighs the lead which is 39 feet 6 inches long, and 3 feet 3 inches broad, at $8\frac{1}{2}$lb. to the fquare foot? Anf. $1091\frac{3}{16}$lb.

EXAM. II. What coft the covering and guttering a roof with lead, at 18s. the cwt.; the length of the roof being 43 feet, and breadth or girt over it 32 feet; the guttering 57 feet long, and 2 feet wide; the former 9·831lb, and the latter 7·373lb. to the fquare foot? Anf. 115l. 9s. $1\frac{1}{2}$d.

XII. TIMBER MEASURING.

PROBLEM I.

To find the Area, or Superficial Content, of a Board or Plank.

MULTIPLY the length by the mean breadth.

Note. When the board is tapering, add the breadths at the two ends together, and take half the fum for the mean breadth.

By the Sliding Rule.

Set 12 on B to the breadth in inches on A; then againft the length in feet on B, is the content on A, in feet and fractional parts.

EXAM-

EXAMPLES.

EXAM. I. What is the value of a plank, at 1½d. per foot, whoſe length is 12 feet 6 inches, and mean breadth 11 inches?

Anſ. 1s. 5d.

EXAM. II. Required the content of a board, whoſe length is 11 feet 2 inches, and breadth 1 foot 10 inches?

Anſ. 20 feet 5 inches 8″.

EXAM. III. What is the value of a plank, which is 12 feet 9 inches long, and 1 foot 3 inches broad, at 2½d. a foot?

Anſ. 3s. 3¼d.

EXAM. IV. Required the value of 5 oaken planks at 3d. per foot, each of them being 17½ feet long; and their ſeveral breadths are as follows, namely, two of 13½ inches in the middle, one of 14½ inches in the middle, and the two remaining ones, each 18 inches at the broader end, and 11¼ at the narrower. Anſ. 1l. 5s. 9½d.

PROBLEM II.

To find the Solid Content of Squared or Four-ſided Timber.

MULTIPLY the mean breadth by the mean thickneſs, and the product again by the length, and the laſt product will give the content.

By the Sliding Rule.

C	D	D	C
As length :	12 or 10 : :	quarter girt :	ſolidity.

That is, as the length in feet on C, is to 12 on D when the quarter girt is in inches, or to 10 on D when it is in tenths of feet; ſo is the quarter girt on D, to the content on C.

Note, 1. If the tree taper regularly from the one end to the other; either take the mean breadth and thickneſs in the middle, or take the dimenſions at the two ends, and half their ſum will be the mean dimenſions.

2. If the piece do not taper regularly, but is unequally thick in ſome parts and ſmall in others, take ſeveral different dimenſions, add them all together, and divide their ſum by the number of them, for the mean dimenſions.

3. The quarter girt is a geometrical mean proportional between the mean breadth and thickneſs, that is the ſquare root of their product. Sometimes unſkilful meaſurers uſe the arithmetical mean inſtead of it, that is half their ſum; but

but this is always attended with error, and the more so as the breadth and depth differ the more from each other.

EXAMPLES.

EXAM. I. The length of a piece of timber is 18 feet 6 inches, the breadths at the greater and less end 1 foot 6 inches and 1 foot 3 inches, and the thickness at the greater and less end 1 foot 3 inches and 1 foot: required the solid content? Ans. 28 feet 7 inches.

EXAM. II. What is the content of the piece of timber, whose length is $24\frac{1}{2}$ feet, and the mean breadth and thickness each 1·04 feet? Ans. $26\frac{1}{2}$ feet.

EXAM. III. Required the content of a piece of timber, whose length is 20·38 feet, and its ends unequal squares, the side of the greater being $19\frac{1}{8}$, and the side of the less $9\frac{7}{8}$? Ans. 29·75625 feet.

EXAM. IV. Required the content of the piece of timber, whose length is 27·36 feet; at the greater end the breadth is 1·78, and thickness 1·23; and at the less end the breadth is 1·04, and thickness 0·91? Ans. 41·278 feet.

PROBLEM III.

To find the Solidity of Round or Unsquared Timber.

MULTIPLY the square of the quarter girt, or of $\frac{1}{4}$ of the mean circumference, by the length, for the content.

By the Sliding Rule.

As the length upon C : 12 or 10 upon D ::
quarter girt, in 12ths or 10ths, on D : content on C.

Note, 1. When the tree is tapering, take the mean dimensions as in the former problems, either by girting it in the middle, for the mean girt, or at the two ends, and take half the sum of the two. But when the tree is very irregular, divide it into several lengths, and find the content of each part separately.

2. This rule, which is commonly used, gives the answer about $\frac{1}{4}$ less than the true quantity in the tree, or nearly what the quantity would be after the tree is hewed square in the usual way: so that it seems intended to make an allowance for the squaring of the tree.

EXAM-

EXAMPLES.

Exam. I. A piece of round timber being 9 feet 6 inches long, and its mean quarter girt 42 inches; what is the content? Anſ. $116\frac{1}{3}$ feet.

Exam. II. The length of a tree is 24 feet, its girt at the thicker end 14 feet, and at the ſmaller end 2 feet; required the content? Anſ. 96 feet.

Exam. III. What is the content of a tree, whoſe mean girt is 3·15 feet, and length 14 feet 6 inches?

Anſ. 8·9922 feet.

Exam. IV. Required the content of a tree, whoſe length is $17\frac{1}{4}$ feet, which girts in five different places as follows, namely, in the firſt place 9·43 feet, in the ſecond 7·92, in the third 6·15, in the fourth 4·74, and in the fifth 3·16?

Anſ. 42·519525.

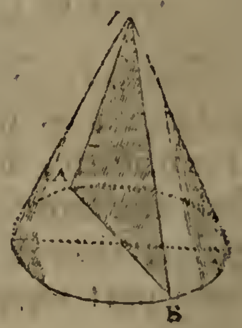
CONIC SECTIONS.

DEFINITIONS.

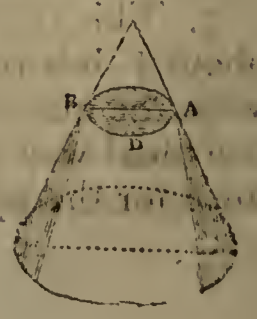
1. Conic Sections are the figures made by the mutual interſection of a cone and a plane.

2. According to the different poſitions of the cutting plane, there ariſe five different figures or ſections, namely, a triangle, a circle, an ellipſe, an hyperbola, and a parabola: the three laſt of which only are peculiarly called Conic Sections.

3. If the cutting plane paſs through the vertex of the cone, and any part of the baſe, the ſection will evidently be a triangle; as VAB.



4. If the plane cut the cone parallel to the baſe, or make no angle with it, the ſection will be a circle; as ABD.



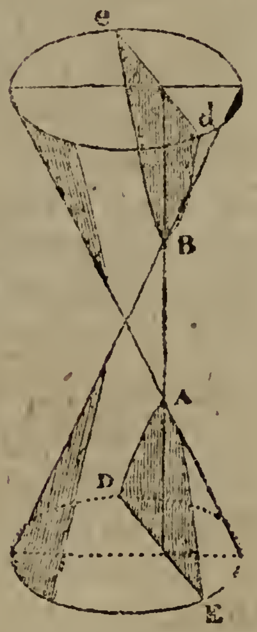
The

5. The ſection DAB is an ellipſe, when the cone is cut obliquely through both ſides, or when the plane is inclined to the baſe in a leſs angle than the ſide of the cone is.

6. The ſection is a parabola, when the cone is cut by a plane parallel to the ſide, or when the cutting plane and the ſide of the cone make equal angles with the baſe.

7. The ſection is an hyperbola, when the cutting plane makes a greater angle with the baſe than the ſide of the cone makes.

8. And if all the ſides of the cone be continued through the vertex, forming an oppoſite equal cone, and the plane be alſo continued to cut the oppoſite cone, this latter ſection will be the oppoſite hyperbola to the former; as dBe.



9. The vertices of any ſection, are the points where the cutting plane meets the oppoſite ſides of the cone, or the ſides of the vertical triangular ſection; as A and B.

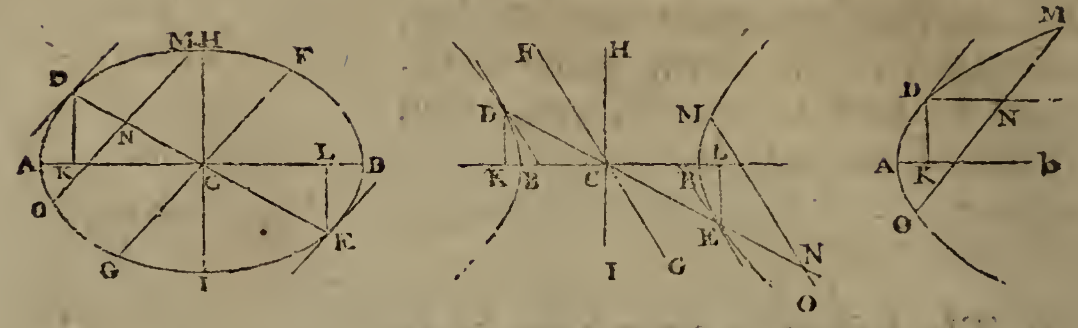
Hence the ellipſe and the oppoſite hyperbolas, have each two vertices; but the parabola only one; unleſs we conſider the other as at an infinite diſtance.

10. The Axis, or Tranſverſe Diameter, of a conic ſection, is the line or diſtance AB between the vertices.

Hence the axis of a parabola is infinite in length, Ab being only a part of it.

Ellipſe.

Ellipſe. Oppoſ. Hyperb. Parabola,



11. The Centre C is the middle of the axis.

Hence the centre of a parabola is infinitely diſtant from the vertex. And of an ellipſe, the axis and centre lie within the curve; but of an hyperbola, without.

12. A Diameter is any right line, as AB or DE, drawn through the centre, and terminated on each ſide by the curve; and the extremities of the diameter, or its interſections with the curve, are its vertices.

Hence all the diameters of a parabola are parallel to the axis, and infinite in length. And hence alſo every diameter of the ellipſe and hyperbola have two vertices; but of the parabola, only one; unleſs we conſider the other as at an infinite diſtance.

13. The Conjugate to any diameter, is the line drawn through the centre, and parallel to the tangent of the curve at the vertex of the diameter. So, FG, parallel to the tangent at D, is the conjugate to DE; and HI, parallel to the tangent at A, is the conjugate to AB.

Hence the conjugate HI, of the axis AB, is perpendicular to it.

14. An Ordinate to any diameter, is a line parallel to its conjugate, or to the tangent at its vertex, and terminated by the diameter and curve. So DK, EL are ordinates to the axis AB; and MN, NO ordinates to the diameter DE.

Hence the ordinates of the axis are perpendicular to it.

15. An Abſciſs is a part of any diameter contained between its vertex and an ordinate to it; as AK or BK, or DN or EN.

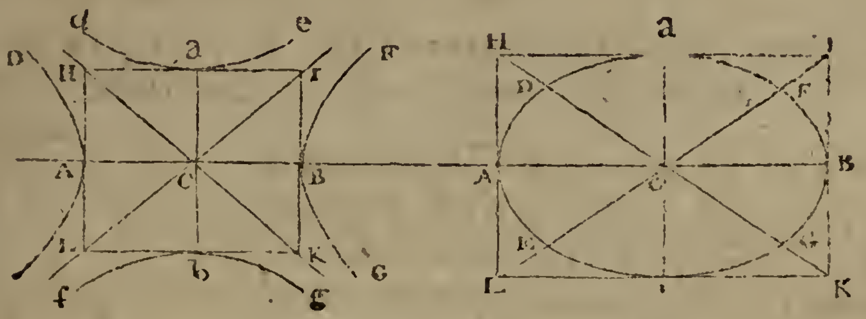
Hence, in the ellipſe and hyperbola, every ordinate has two abſciſſes; but in the parabola, only one; the other vertex of the diameter being infinitely diſtant.

16. The Parameter of any diameter, is a third proportional to that diameter and its conjugate.

17. The

17. The Focus is the point in the axis where the ordinate is equal to half the parameter. As K and L, where DK or EL is equal to the femi-parameter.

Hence, the ellipfe and hyperbola have each two foci; but the parabola only one.



18. If DAE, FBG be two oppofite hyperbolas, having AB for their firft or tranfverfe axis, and ab for their fecond or conjugate axis. And if dae, fbg be two other oppofite hyperbolas having the fame axes, but in the contrary order, namely, ab their firft axis, and AB their fecond; then thefe two latter curves dae, fbg, are called the conjugate hyperbolas to the two former DAE, FBG; and each pair of oppofite curves mutually conjugate to the other.

19. And if tangents be drawn to the four vertices of the curves, or extremities of the axes, forming the infcribed rectangle HIKL; the diagonals HCK, ICL of this rectangle, are called the afymptotes of the curves. And if thefe afymptotes interfect at right angles, or the infcribed rectangle be a fquare, or the two axes AB and ab be equal, then the hyperbolas are faid to be right-angled, or equilateral.

SCHOLIUM.

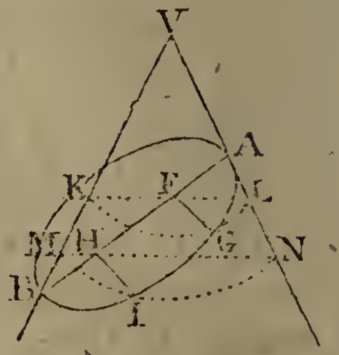
The rectangle infcribed between the four conjugate hyperbolas, is fimilar to a rectangle circumfcribed about an ellipfe, by drawing tangents, in like manner, to the four extremities of the two axes; and the afymptotes or diagonals in the hyperbola, are analagous to thofe in the ellipfe, cutting this curve in fimilar points, and making the pair of equal conjugate diameters. Moreover, the whole figure, formed by the four hyperbolas, is, as it were, an ellipfe turned infide out, cut open at the extremities D, E, F, G, of the faid equal conjugate diameters, and thofe four points drawn out to an infinite diftance, the curvature being turned the contrary way, but the axes, and the rectangle paffing through their extremities, continuing fixed.

Of

OF THE ELLIPSE.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



LET AVB be a plane passing through the axis of the cone; AGIH another section of the cone perpendicular to the plane of the former; AB the axis of this elliptic section; and FG, HI ordinates perpendicular to it. Then it will be, as FG : HI :: AF . FB : AH . HB.

For, through the ordinates FG, HI draw the circular sections KGL, MIN parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the ellipse.

Now, by the similar triangles AFL, AHN, and BFK, BHM,

it is AF : AH :: FL : HN,

and FB : HB :: KF : MH;

hence, taking the rectangles of the corresponding terms,

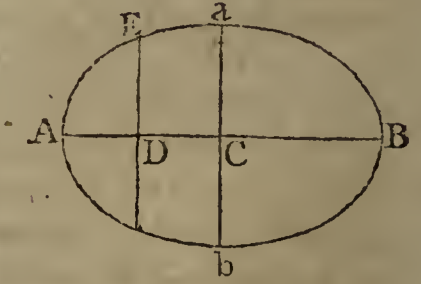
it is, the rect. AF . FB : AH . HB :: KF . FL : MH . HN.

But, by the circle, KF . FL $=$ FG^2, and MH . HN $=$ HI^2;

Therefore the rect. AF . FB : AH . HB :: FG^2 : HI^2. Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abscisses :
To the Square of their Ordinate.



That is, AB^2 : ab^2 or AC^2 : ac^2 :: AD.DB : DE^2.

For, by theor. 1, AC . CB : AD . DB :: ca^2 : DE^2;

But, if C be the centre, then AC . CB $=$ AC^2, and ca is the semi-conj.

Therefore

Therefore $AC^2 : AD . DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD . DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD . DB : DE^2$. Q. E. D.

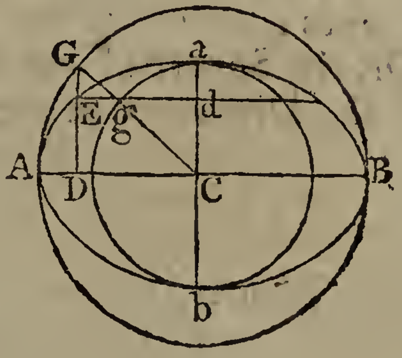
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: CA^2 - CD^2 : DE^2$,
that is, $AB : p :: AD . DB$ or $CA^2 - CD^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the tranſverſe,
Is to its parameter,
So is the rectangle of the abſciſſes,
To the ſquare of their ordinate.

THEOREM III.

As the Square of the Conjugate Axis :
Is to the Square of the Tranſverſe Axis ::
So is the Rectangle of the Abſciſſes of the Conjugate, or the Difference of the Squares of the Semi-conjugate and Diſtance of the Centre from any Ordinate of that Axis :
To the Square of their Ordinate.

That is,
$ca^2 : CB^2 :: ad . db$ or $ca^2 - cd^2 : dE^2$.



For, draw the ordinate ED to the tranſverſe AB.
Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD . DB$ or $CA^2 - CD^2$,
or - - $ca^2 : CA^2 :: cd^2 : CA^2 - dE^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$,
theref. by diviſion, $ca^2 : CA^2 :: ca^2 - cd^2$ or $ad . db : dE^2$.
Q. E. D.

Corol. 1. If two circles be deſcribed on the two axes as diameters, the one inſcribed within the ellipſe, and the other circumſcribed about it; then an ordinate in the circle will be to the correſponding ordinate in the ellipſe, as the axis of this ordinate, is to the other axis.

That is, $CA : ca :: DG : DE$,
and $ca : CA :: dg : dE$.

 For,

For, by the nature of the circle, $AD \cdot DB = DG^2$; theref. by the nature of the ellipse, $CA^2 : Ca^2 :: AD \cdot DB$ or $DG^2 : DE^2$,

or $CA : ca :: DG : DE$.

In like manner - $ca : CA :: dg : dE$.

Moreover, by equality, $DG : DE$ or $Cd :: dE$ or $DC : dg$.

Therefore CgG is a continued straight line.

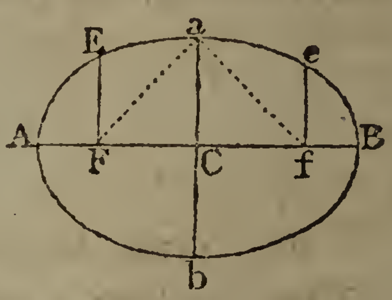
Corol. 2. Hence also, as the ellipse and circle are made up of the same number of corresponding ordinates, which are all in the same proportion of the two axes, it follows that the areas of the whole circle and ellipse, as also of any like parts of them, are in the same proportion of the two axes, or as the square of the diameter to the rectangle of the two axes; that is, the areas of the two circles, and of the ellipse, are as the square of each axis and the rectangle of the two, and therefore the ellipse is a mean proportional between the two circles.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Difference of the Squares of the Semi-axes;

Or, the Square of the Distance between the Foci, is equal to the Difference of the Squares of the two Axes.

That is, $CF^2 = CA^2 - Ca^2$,
or $Ff^2 = AB^2 - ab^2$.



For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature of the curve - $CA^2 : Ca^2 :: CA^2 - CF^2 : FE^2$;
and by the def. of the para. $CA^2 : Ca^2 :: Ca^2 : FE^2$;
therefore - - $Ca^2 = CA^2 - CF^2$;
and by addit. and subtr. $CF^2 = CA^2 - Ca^2$;
or, by doubling, - $Ff^2 = AB^2 - ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CFa; and the distance Fa from the focus to the extremity of the conjugate axis, is $= AC$ the semi-transverse.

Corol. 2.

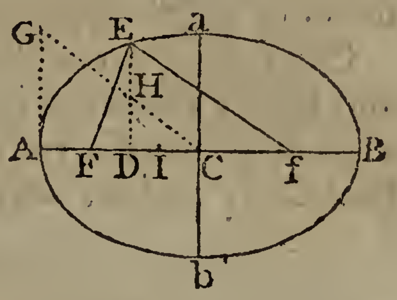
Corol. 2. The conjugate ſemi-axis Ca is a mean proportional between AF, FB, or between Af, fB, the diſtances of either focus from the two vertices.

For $Ca^2 = CA^2 - CF^2 = CA + CF \cdot CA - CF = AF \cdot FB$.

THEOREM V.

The Sum of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Tranſverſe Axis.

That is,
$FE + fE = AB$.



For, draw AG parallel and equal to Ca the ſemi-conjugate; and join CG meeting the ordinate DE in H; alſo take CI a 4th proportional to CA, CF, CD.

Then, by theor. 2, $CA^2 : AG^2 :: CA^2 - CD^2 : DE^2$;
and, by ſim. tri. $CA^2 : AG^2 :: CA^2 - CD^2 : AG^2 - DH^2$;
conſequently $DE^2 = AG^2 - DH^2 = Ca^2 - DH^2$.

Alſo $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \cdot CD + CD^2$;
And, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 + Ca^2 - 2CF \cdot CD + CD^2 - DH^2$.

But by theor. 4, $CF^2 + Ca^2 = CA^2$,
and, by ſuppoſition, $2CF \cdot CD = 2CA \cdot CI$;
theref. $FE^2 = CA^2 - 2CA \cdot CI + CD^2 - DH^2$.

Again, by ſupp. $CA^2 : CD^2 :: CF^2$ or $CA^2 - AG^2 : CI^2$;
and, by ſim. tri. $CA^2 : CD^2 :: CA^2 - AG^2 : CD^2 - DH^2$;
therefore - $CI^2 = CD^2 - DH^2$;
conſequently $FE^2 = CA^2 - 2CA \cdot CI + CI^2$.

And the root or ſide of this ſquare is $FE = CA - CI = AI$.

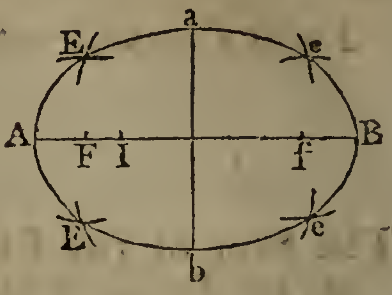
In the ſame manner it is found that $fE = CA + CI = BI$.
Conſeq. by addit. $FE + fE = AI + BI = AB$. Q. E. D.

Corol. 1. Hence CI or $CA - FE$ is a 4th proportional to CA, CF, CD.

Corol. 2. And $fE - FE = 2CI$; that is, the difference between two lines drawn from the foci, to any point in the curve, is double the 4th proportional to CA, CF, CD.

H 2 *Corol.* 3.

Corol. 3. Hence is derived the common method of describing this curve mechanically by points, or with a thread, thus:



In the transverse take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, describe arcs intersecting in E, which will be a point in the curve. In like manner, assuming other points i, as many other points will be found in the curve. Then with a steady hand, the curve line may be drawn through all the points of intersection E.

Or, take a thread of the length of AB the transverse axis, and fix its two ends in the foci F, f, by two pins. Then carry a pen or pencil round by the thread, keeping it always stretched, and its point will trace out the curve line.

THEOREM VI.

If from any Point I in the Axis produced, a Line IL be drawn touching the curve in one Point L; and the Ordinate LM be drawn; and if C be the Centre or Middle of AB: Then shall CM be to CI as the Square of AM to the Square of AI.

That is,
$CM : CI :: AM^2 : AI^2$.

For, from the point I draw any other line IEH to cut the curve in two points E and H; from which let fall the perpendiculars ED and HG; and bisect DG in K.

Then, by theor. 1, $AD \cdot DB : AG \cdot GB :: DE^2 : GH^2$,
and by sim. tri. $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, $AD \cdot DB : AG \cdot GB :: ID^2 : IG^2$.

But $DB = CB + CD = AC + CD = AG + DC - CG = 2CK + AG$,
and $GB = CB - CG = AC - CG = AD + DC - CG = 2CK + AD$;
theref. $AD \cdot 2CK + AD \cdot AG : AG \cdot 2CK + AD \cdot AG :: ID^2 : IG^2$,
and, by div. $DG \cdot 2CK : IG^2 - ID^2$ or $DG \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$.
or - $2CK : 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$,
or - $AD \cdot 2CK : AD \cdot 2IK :: AD \cdot 2CK + AD \cdot AG : ID^2$;
theref. by div. $CK : IK :: AD \cdot AG : ID^2 - AD \cdot 2IK$,
and, by comp. $CK : CI :: AD \cdot AG : ID^2 - AD \cdot \overline{ID + IA}$,
or - $CK : CI :: AD \cdot AG : AI^2$.

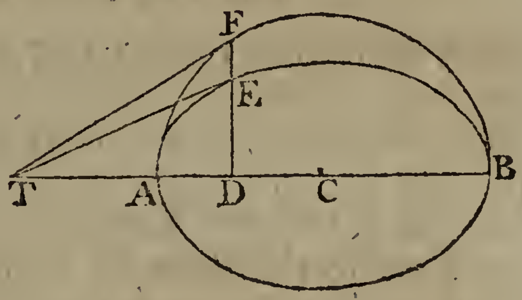
But,

But, when the line IH, by revolving about the point I, comes into the poſition of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the laſt proportion becomes CM : CI :: $AM^2 : AI^2$.

THEOREM VII.

If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Tranſverſe Axis; the Semi-tranſverſe will be a Mean Proportional between the Diſtances of the ſaid Two Interſections from the Centre.

That is,
CA is a mean proportional between CD and CT; or CD, CA, CT are continued proportionals.



For, by theor. 6, CD : CT :: $AD^2 : AT^2$,
that is, CD : CT :: $(CA - CD)^2 : (CT - CA)^2$,
or - CD : CT :: $CD^2 + CA^2 : CA^2 + CT^2$,
and - CD : DT :: $CD^2 + CA^2 : CT^2 - CD^2$,
or - CD : DT :: $GD^2 + CA^2 : (CT + CD)\,DT$,
or - $CD^2 : CD.CT :: CD^2 + CA^2 : CD.DT + CT.DT$,
hence - $CD^2 : CA^2 :: CD.DT : CT.DT$,
and - $CD^2 : CA^2 ::$ CD : CT.
therefore (th. 78, Geom.) CD : CA :: CA : CT. Q. E. D.

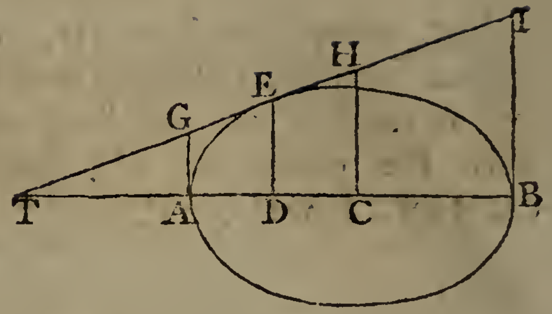
Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain conſtant, then will the point T be conſtant alſo; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every ellipſe deſcribed on the ſame axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from theſe four points, namely the Centre, the two Extremities of the Axis, and the Point of Contact; thoſe Four Perpendiculars will be Proportionals.

That

That is,
AG : DE :: CH : BI.



For, by theor. 7, TC : AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB,
and by comp. TA : TD :: TC : TB,
and by fim. tri. AG : DE :: CH : BI. Q. E. D.

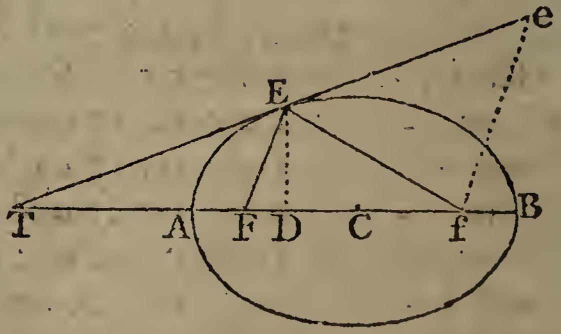
Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI } are alfo proportionals.

For thefe are as AG, DE, CH, BI, by fimilar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; thefe two Lines will make equal Angles with the Tangent.

That is,
the $\angle$ FET $= \angle$ fEe.



For, draw the ordinate DE, and fe parallel to FE.
By cor. 1, theor. 5, CA : CD :: CF : CA — FE,
and by theor. 7, CA : CD :: CT : CA;
therefore CT : CF :: CA : CA — FE;
and by add. and fub. TF : Tf :: FE : 2CA — FE or fE by th. 5,
But by fim. tri. TF : Tf :: FE : fe;
therefore fE = fe, and confeq. $\angle$e = $\angle$fEe.
But, becaufe FE is parallel to fe, the $\angle$e = $\angle$FET;
therefore the $\angle$ FET = $\angle$ fEe. Q. E. D.

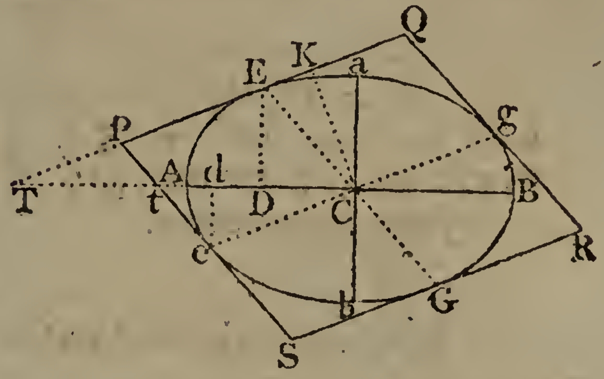
Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this theorem, that rays of light iffuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from thofe points to the other focus. So the ray fE is reflected into FE. And this is the reafon why the points F, f are called *foci*, or burning points.

THE-

THEOREM X.

All the Parallelograms circumscribed about an Ellipse are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS = the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four lesser and equal parallelograms. Also, draw the ordinates DE, dc, and CK perpendicular to PQ; and let the axis CA produced meet the sides of the parallelogram, produced if necessary, in T and t.

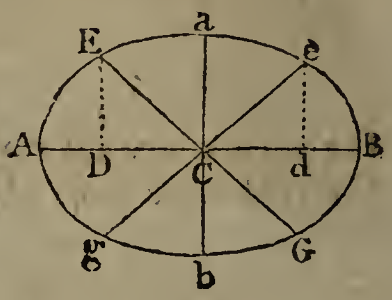
Then, by theor. 7, CT : CA :: CA : CD,
and ct : CA :: CA : cd ;
theref. by equality, CT : ct :: cd : CD ;
but, by sim. triangles, CT : ct :: TD : cd,
theref. by equality, TD : cd :: cd : CD,
and the rectangle TD . DC is = the square cd^2.
Again, by theor. 7, CD : CA :: CA : CT,
or, by division, CD : CA :: DA : AT,
and by composition, CD : DB :: AD : DT ;
conseq. the rectangle CD . DT = cd^2 = AD . DB*.
But, by theor. 1, $CA^2 : ca^2$ :: (AD . DB or) $cd^2 : DE^2$,
therefore - CA : ca :: cd : DE ;
In like manner, CA : ca :: CD : de,
or - - ca : de :: CA : CD.
But, by theor. 7, CT : CA :: CA : CD ;
theref. by equality, CT : CA :: ca : de.
But, by sim. tri. CT : CK :: ce : de ;
theref. by equality, CK : CA :: ca : ce,
and the rectangle CK . ce = CA . ca.
But the rect. CK . ce = the parallelogram CEPc,
theref. the rect. CA . ca = the parallelogram CEPe,
and conseq. the rect. AB . ab = the paral. PQRS. Q. E. D.

* *Corol.* Because cd^2 = AD . DB = $CA^2 - CD^2$,
therefore $CA^2 = CD^2 + cd^2$.
In like manner, $ca^2 = DE^2 + dc^2$.

THE

THEOREM XI.

The Sum of the Squares of every Pair of Conjugate Diameters, is equal to the same constant Quantity, namely, the Sum of the Squares of the two Axes.



That is,
$AB^2 + ab^2 = EG^2 + eg^2$;
where EG, eg are any pair of conjugate diameters.

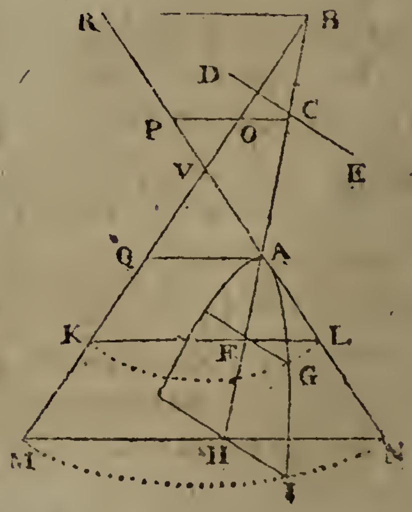
For, draw the ordinates ED, ed.

Then, by cor. to theor. 10, $CA^2 = CD^2 + cd^2$,
and - - - $ca^2 = DE^2 + de^2$;
therefore the sum $CA^2 + ca^2 = CD^2 + DE^2 + cd^2 + de^2$.
But, by right-angled $\triangle$s, $CE^2 = CD^2 + DE^2$,
and - - $ce^2 = cd^2 + de^2$;
therefore the sum $CE^2 + ce^2 = CD^2 + DE^2 + cd^2 + de^2$.
consequently - $CA^2 + ca^2 = CE^2 + ce^2$;
or, by doubling, $AB^2 + ab^2 = EG^2 + eg^2$. Q. E. D.

OF THE HYPERBOLA.

THEOREM I.

The Squares of the Ordinates of the Axis are to each other as the Rectangles of their Abscisses.



LET AVB be a plane passing through the vertex and axis of the opposite cones; AGIH another section of them perpendicular to the plane of the former; AB the axis of the hyperbolic sections; and FG, HI ordinates perpendicular to it. Then it will be, as
$FG^2 : HI^2 :: AF \cdot FB : AH \cdot HB$.

For, through the ordinates FG, HI draw the circular sections KGL, MIN

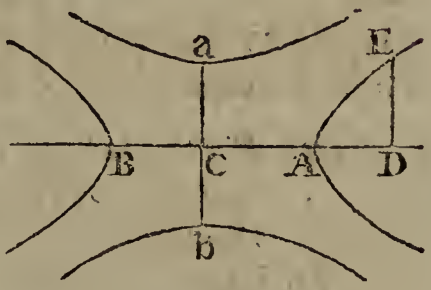
MIN parallel to the base of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the hyperbola.

Now, by the similar triangles AFL, AHN, and BFK, BHM,
it is $AF : AH :: FL : HN$,
and $FB : HB :: KF : MH$;
hence, taking the rectangles of the corresponding terms,
it is, the rect. $AF \cdot FB : AH \cdot HB :: KF \cdot FL : MH \cdot HN$.
But, by the circle, $KF \cdot FL = FG^2$, and $MH \cdot HN = HI^2$;
Therefore the rect. $AF \cdot FB : AH \cdot HB :: FG^2 : HI^2$.

Q. E. D.

THEOREM II.

As the Square of the Transverse Axis :
Is to the Square of the Conjugate ::
So is the Rectangle of the Abscisses :
To the Square of their Ordinate.



That is, $AB^2 : ab^2$ or
$AC^2 : ac^2 :: AD \cdot DB : DE^2$.

For, by theor, I, $AC \cdot CB : AD \cdot DB :: Ca^2 : DE^2$;
But, if C be the centre, then $AC \cdot CB = AC^2$, and Ca is the semi-conj.
Therefore - $AC^2 : AD \cdot DB :: ac^2 : DE^2$;
or, by permutation, $AC^2 : ac^2 :: AD \cdot DB : DE^2$;
or, by doubling, $AB^2 : ab^2 :: AD \cdot DB : DE^2$. Q. E. D.

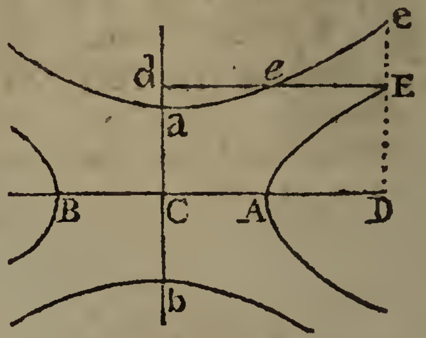
Corol. Or, by div. $AB : \frac{ab^2}{AB} :: CD^2 - CA^2 : DE^2$,
that is, $AB : p :: AD \cdot DB$ or $CD^2 - CA^2 : DE^2$;
where p is the parameter $\frac{ab^2}{AB}$ by the definition of it.

That is, As the transverse,
Is to its parameter,
So is the rectangle of the abscisses,
To the square of their ordinate.

THE-

THEOREM III.

As the Square of the Conjugate Axis :
To the Square of the Transverse Axis : :
So is the Sum of the Squares of the Semi-conjugate, and Distance of the Centre from any Ordinate of this Axis :
To the Square of their Ordinate.



That is,
$ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.

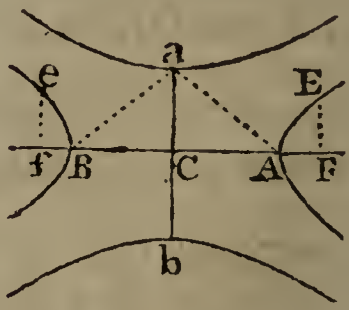
For, draw the ordinate ED to the transverse AB.

Then, by theor. 1, $ca^2 : CA^2 :: DE^2 : AD . DB$ or $CD^2 - CA^2$,
or - - $ca^2 : CA^2 :: cd^2 : dE^2 - CA^2$.
But - - $ca^2 : CA^2 :: ca^2 : CA^2$,
theref. by compos. $ca^2 : CA^2 :: ca^2 + cd^2 : dE^2$.
In like manner, $CA^2 : ca^2 :: CA^2 + CD^2 : De^2$. Q. E. D.

Corol. By the last theor. $CA^2 : ca^2 :: CD^2 - CA^2 : DE^2$,
and by this theor. $CA^2 : ca^2 :: CD^2 + CA^2 : De^2$,
therefore - $DE^2 : De^2 :: CD^2 - CA^2 : CD^2 + CA^2$.
In like manner, $de^2 : dE^2 :: cd^2 - ca^2 : cd^2 + ca^2$.

THEOREM IV.

The Square of the Distance of the Focus from the Centre, is equal to the Sum of the Squares of the Semi-Axes.
Or, the Square of the Distance between the Foci, is equal to the Sum of the Squares of the two Axes.



That is,
$CF^2 = CA^2 + ca^2$, or
$Ff^2 = AB^2 + ab^2$.

For, to the focus F draw the ordinate FE; which, by the definition, will be the semi-parameter. Then, by the nature

nature of the curve - $CA^2 : ca^2 :: CF^2 - CA^2 : FE^2$;
and by the def. of the para. $CA^2 : ca^2 :: ca^2 : FE^2$;
therefore - - $ca^2 = CF^2 - CA^2$;
and by addition, - $CF^2 = CA^2 + ca^2$;
or, by doubling, - $Ff^2 = AB^2 + ab^2$. Q. E. D.

Corol. 1. The two semi-axes, and the focal distance from the centre, are the sides of a right-angled triangle CAa; and the distance Aa is $= CF$ the focal distance.

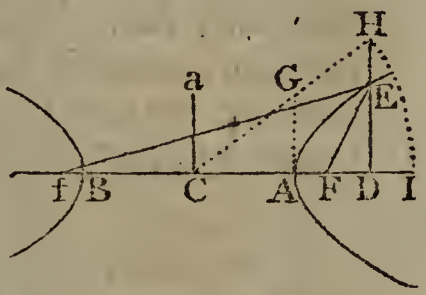
Corol. 2. The conjugate semi-axis ca is a mean proportional between AF, FB, or between Af, fB, the distances of either focus from the two vertices.

For $ca^2 = CF^2 - CA^2 = CF + CA \,.\, CF - CA = AF \,.\, FB$.

THEOREM V.

The Difference of two Lines drawn from the two Foci, to meet at any Point in the Curve, is equal to the Transverse Axis.

That is,
$fE - FE = AB$.



FOR, draw AG parallel and equal to ca the semi-conjugate; and join CG meeting the ordinate DE produced in H; also take CI a 4th proportional to CA, CF, CD.

Then, by th. 2, $CA^2 : AG^2 :: CD^2 - CA^2 : DE^2$;
and, by sim. $\triangle$s, $CA^2 : AG^2 :: CD^2 - CA^2 : DH^2 - AG^2$;
consequently $DE^2 = DH^2 - AG^2 = DH^2 - ca^2$.
Also, $FD = CF \backsim CD$, and $FD^2 = CF^2 - 2CF \,.\, CD + CD^2$;
and, by right-angled triangles, $FE^2 = FD^2 + DE^2$;
therefore $FE^2 = CF^2 - ca^2 - 2CF \,.\, CD + CD^2 + DH^2$.
But, by theor. 4, $CF^2 - ca^2 = CA^2$,
and, by supposition, $2CF \,.\, CD = 2CA \,.\, CI$;
theref. $FE^2 = CA^2 - 2CA \,.\, CI + CD^2 + DH^2$;
Again, by suppos. $CA^2 : CD^2 :: CF^2$ or $CA^2 + AG^2 : CI^2$;
and, by sim. tri. $CA^2 : CD^2 :: CA^2 + AG^2 : CD^2 + DH^2$;
therefore - $CI^2 = CD^2 + DH^2 = CH^2$;
consequently $FE^2 = CA^2 - 2CA \,.\, CI + CI^2$.
And the root or side of this square is $FE = CI - CA = AI$.
In the same manner, it is found that $fE = CI + CA = BI$.
Conseq. by subtract. $fE - FE = BI - AI = AB$. Q. E. D.

Corol. 1.

Corol. 1. Hence CH = CI is a 4th proportional to CA, CF, CD.

Corol. 2. And fE + FE = 2CH or 2CI; or FE, CH, fE are in continued arithmetical progreſſion, the common difference being CA the ſemi-tranſverſe.

Corol. 3. Hence is derived the common method of deſcribing this curve mechanically by points, thus:

In the tranſverſe AB, produced, take the foci F, f, and any point I. Then with the radii AI, BI, and centres F, f, deſcribe arcs interſecting in E, which will be a point in the curve. In like manner, aſſuming other points I, as many other points will be found in the curve.

Then with a ſteady hand, the curve line may be drawn through all the points of interſection E.

In the ſame manner are conſtructed the other two hyperbolas, uſing the axis ab inſtead of AB.

THEOREM VI.

If from any Point I in the Axis, a Line IL be drawn touching the Curve in one Point L; and the ordinate LM be drawn; and if C be the Centre or the Middle of AB: Then ſhall CM be to CI as the Square of AM to the Square of AI.

That is,
$CM : CI :: AM^2 : AI^2$.

For, from the point I draw any line AEH to cut the curve in two points E and H; from which let fall the perps. ED, HG; and biſect DG in K.

Then, by theor. 1, $AD . DB : AG . GB :: DE^2 : GH^2$,
and by ſim. tri. - $ID^2 : IG^2 :: DE^2 : GH^2$;
theref. by equality, - $AD . DB : AG . GB :: ID^2 : IG^2$,
But $DB = CB + CD = CA + CD = CG + CD - AG = 2CK - AG$,
and $GB = CB + CG = CA + CG = CG + CD - AD = 2CK - AD$;
theref. $AD . 2CK - AD . AG : AG . 2CK - AD . AG :: ID^2 : IG^2$,
and, by div. $DG . 2CK : IG^2 - ID^2$ or $DG . 2IK :: AD . 2CK - AD . AG : ID^2$.
or - $2CK : 2IK :: AD . 2CK - AD . AG : ID^2$;
or $AD . 2CK : AD . 2IK :: AD . 2CK - AD . AG : ID^2$;
theref. by div. $CK : IK :: AD . AG : AD . 2IK - ID^2$,
and, by div. $CK : CI :: AD . AG : ID^2 - AD . \overline{ID + IA}$,
or - $CK : CI :: AD . AG : AI^2$.

But,

But, when the line IH, by revolving about the point I, comes into the poſition of the tangent IL, then the points E and H meet in the point L, and the points D, K, G, coincide with the point M; and then the laſt proportion becomes $CM : CI :: AM^2 : AI^2$. Q. E. D.

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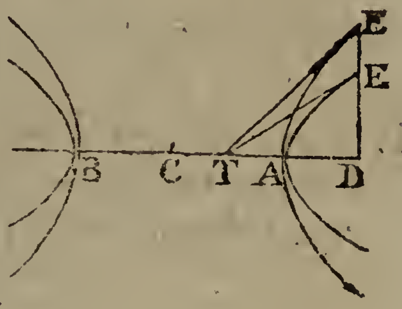
If a Tangent and Ordinate be drawn from any Point in the Curve, meeting the Tranſverſe Axis; the Semi-tranſverſe will be a Mean Proportional between the Diſtances of the ſaid Two Interſections from the Centre.

That is,
CA is a mean proportion between CD and CT; or CD, CA, CT are continued proportionals.



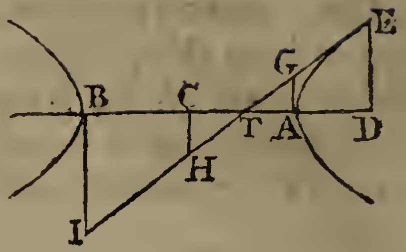
FOR, by th. 6, $CD : CT :: AD^2 : AT^2$,
that is, - $CD : CT :: (CD - CA)^2 : (CA - CT)^2$,
or - - $CD : CT :: CD^2 + CA^2 : CA^2 + CT^2$,
and - - $CD : DT :: CD^2 + CA^2 : CD^2 - CT^2$,
or - - $CD : DT :: CD^2 + CA^2 : (CD + CT)\,DT$,
or, $CD^2 : CD . DT :: CD^2 + CA^2 : CD . DT + CT . TD$;
hence $CD^2 : CA^2 :: CD . DT : CT . TD$,
and $CD^2 : CA^2 :: CD : CT$,
theref. (th. 78, Geom.) $CD : CA :: CA : CD$. Q. E. D.

Corol. Since CT is always a third proportional to CD, CA; if the points D, A, remain conſtant, then will the point T be conſtant alſo; and therefore all the tangents will meet in this point T, which are drawn from the point E, of every hyperbola deſcribed on the ſame axis AB, where they are cut by the common ordinate DEE drawn from the point D.

THEOREM VIII.

If there be any Tangent meeting Four Perpendiculars to the Axis drawn from theſe four Points, namely, the Centre, the two Extremities of the Axis, and the Point of Contact; thoſe Four Perpendiculars will be Proportionals.

That



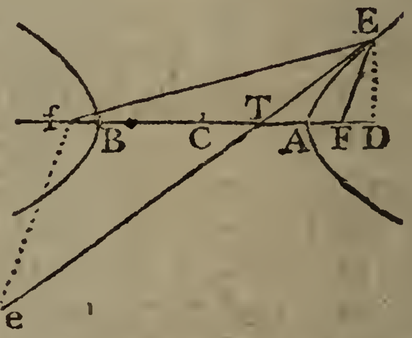
That is,
AG : DE :: CH : BI.

For, by theor. 7, TC : AC :: AC : DC,
theref. by div. TA : AD :: TC : AC or CB;
and by comp. TA : TD :: TC : TB;
and by fim. tri. AG : DE :: CH : BI. Q. E. D.

Corol. Hence TA, TD, TC, TB and TG, TE, TH, TI } are alfo proportionals.
For thefe are as AG, DE, CH, BI, by fimilar triangles.

THEOREM IX.

If there be any Tangent, and two Lines drawn from the Foci to the Point of Contact; thefe two Lines will make equal Angles with the Tangent.



That is,
the ∠ FET = ∠ fEC.

For, draw the ordinate DE, and fe parallel to FE.

By cor. 1, theor. 5, CA : CD :: CF : CA + FE;
and by th. 7, CA : CD :: CT : CA;
therefore - CT : CF :: CA : CA + FE;
and by add. and fub. TF : Tf :: FE : 2CA + FE or fE by th. 5.
But by fim. tri. TF : Tf :: FE : fe;
therefore - fE = fe, and confeq. ∠ e = ∠ fEc.
But, becaufe FE is parallel to fe, the ∠ e = ∠ FET;
therefore the ∠ FET = ∠ fEc. Q. E. D.

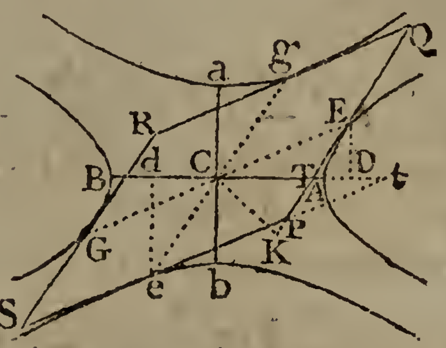
Corol. As opticians find that the angle of incidence is equal to the angle of reflection, it appears from this propofition, that rays of light iffuing from the one focus, and meeting the curve in every point, will be reflected into lines drawn from the other focus. So the ray fE is reflected into FE. And this is the reafon why the points F, f are called *foci*, or burning points.

THE-

THEOREM X.

All the Parallelograms inscribed between the four Conjugate Hyperbolas are equal to one another, and each equal to the Rectangle of the two Axes.

That is,
the parallelogram PQRS =
the rectangle AB . ab.



Let EG, eg be two conjugate diameters parallel to the sides of the parallelogram, and dividing it into four lesser and equal parallelograms. Also, draw the ordinates DE, de, and CK perpendicular to PQ; and let the axis produced meet the sides of the parallelogram, produced, if necessary, in T and t.

Then, by theor. 7,	CT : CA :: CA : CD,
and	ct : CA :: CA : cd;
theref. by equality,	CT : ct :: cd : CD;
but, by sim. triangles,	CT : ct :: TD : cd,
theref. by equality,	TD : cd :: cd : CD,
and the rectangle	TD . DC is = the square cd^2.
Again, by theor. 7,	CD : CA :: CA : CT,
or, by division	CD : CA :: DA : AT,
and, by composition,	CD : DB :: DA : DT;
conseq. the rectangle	CD . DT = cd^2 = AD . DB *.
But, by theor. 1,	$CA^2 : ca^2$ :: (AD . DB or) $cd^2 : DE^2$,
therefore -	CA : ca :: cd : DE;
In like manner,	CA : ca :: CD : de;
or - - -	ca : de :: CA : CD.
But, by theor. 7,	CT : CA :: CA : CD;
theref. by equality,	CT : CA :: ca : de.
But, by sim. tri.	CT : CK :: ce : de;
theref. by equality,	CK : CA :: ca : ce,
and the rectangle	CK . ce = CA . ca.
But the rect. -	CK . ce = the parallelogram CEPe,
theref. the rect.	CA . ce = the parallelogram CEPe,
and conseq. the rect.	AB . ab = the paral. PQRS. Q. E. D.

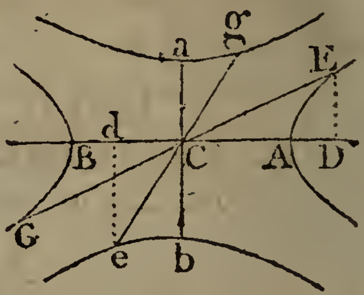
* *Corol.* Because cd^2 = AD . DB = $CD^2 - CA^2$,
therefore $CA^2 = CD^2 - cd^2$.
In like manner, $ca^2 = de^2 - DE^2$.

THE-

THEOREM XI.

The Difference of the Squares of every Pair of Conjugate Diameters, is equal to the ſame conſtant Quantity, namely the Difference of the Squares of the two Axes.

That is,
$AB^2 - ab^2 = EG^2 - eg^2$;
where EG, eg are any conjugate diameters.



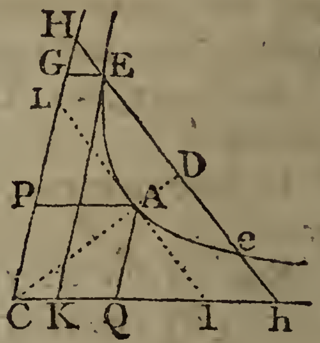
For, draw the ordinates ED, ed.

Then, by cor. to theor. 10, $CA^2 = CD^2 - cd^2$,
and - - $ca^2 = de^2 - DE^2$;
theref. the difference $CA^2 - ca^2 = CD^2 + DE^2 - cd^2 - de^2$.
But, by right-angled $\triangle$s, $CE^2 = CD^2 + DE^2$,
and - - $ce^2 = cd^2 + de^2$;
theref. the difference $CE^2 - ce^2 = CD^2 + DE^2 - cd^2 - de^2$.
conſequently - $CA^2 - ca^2 = CE^2 - ce^2$;
or, by doubling, $AB^2 - ab^2 = EG^2 - eg^2$. Q. E. D.

THEOREM XII.

All the Parallelograms are equal which are formed between the Aſymptotes and Curve, by Lines drawn Parallel to the Aſymptotes.

That is, the lines GE, EK, AP, AQ being parallel to the aſymptotes CH, Cl; then the paral. CGEK = paral. CPAQ.



For, let A be the vertex of the curve, or extremity of the ſemi-tranſverſe axis AC, perp. to which draw AL or Al, which will be equal to the ſemi-conjugate, by definition 19. Alſo, draw HEDeh parallel to Ll,

Then, by theor. 2, $CA^2 : AL^2 :: CD^2 - CA^2 : DE^2$,
and, by parallels, $CA^2 : AL^2 :: CD^2 : DH^2$;
theref. by ſubtract, $CA^2 : AL^2 :: CA^2 : DH^2 - DE^2$ or rect. HE . Eh;
conſeq. the ſquare AL^2 = the rect. HE . Eh.

But,

But, by sim. tri. PA : AL :: GE : EH,
and, by the same, QA : Al :: EK : Eh;
theref. by comp. PA . AQ : AL^2 :: GE . EK : HE . Eh;
and, because AL^2 = HE . Eh, theref. PA . AQ = GE . EK.

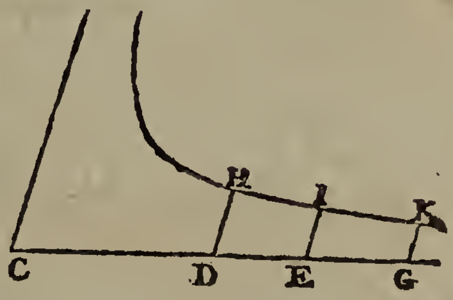
But the parallelograms CGEK, CPAQ, being equiangular, are as the rectangles GE . EK and PA . AQ.

Therefore the parallelogram GK = the paral. PQ.

That is, all the inscribed parallelograms are equal to one another. Q. E. D.

Corol. 1, Because the rectangle GEK or CGE is constant, therefore GE is reciprocally as CG, or CG : CP :: PA : GE. And hence the asymptote continually approaches towards the curve, but never meets it: for GE decreases continually as CG increases; and it is always of *some* magnitude, except when CG is supposed to be infinitely great, for then GE is infinitely small, or nothing. So that the asymptote CG may be considered as a tangent to the curve at a point infinitely distant from C.

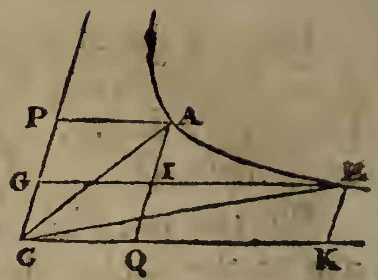
Corol. 2. If the abscisses CD, CE, CG, &c, taken on the one asymptote, be in geometrical progression increasing; then shall the ordinates DH, EI, GK, &c, parallel to the other asymptote, be a decreasing geometrical progression, having the same ratio. For, all the rectangles CDH, CEI, CGK, &c, being equal, the ordinates DH, EI, GK, &c, are reciprocally as the abscisses CD, CE, CG, &c, which are geometricals. And the reciprocals of geometricals are also geometricals, and in the same ratio, but decreasing, or in converse order.



THEOREM XIII.

The three following Spaces, between the Asymptotes and the Curve, are equal; namely, the Sector or Trilinear Space contained by an Arc of the Curve and two Radii, or Lines drawn from its Extremities to the Centre; and each of the two Quadrilaterals, contained by the said Arc, and two Lines drawn from its Extremities parallel to one Asymptote, and the intercepted Part of the other Asymptote.

 That



That is,
The fector CAE = PAEG = QAEK, all ftanding on the fame arc AE.

For, by theor. 12, CPAQ = CGEK; fubtract the common fpace CGIQ, there remains the paral. PI = the par. IK; to each add the trilineal IAE, the fum is the quadr. PAEG = QAEK.

Again, from the quadrilateral CAEK take the equal triangles CAQ, CEK, and there remains the fector CAE = QAEK. Therefore CAE = QAEK = PAEG. Q. E. D.

Of the PARABOLA.

THEOREM I.

The Abfciffes are Proportional to the Squares of their Ordinates.

LET AVM be a fection through the axis of the cone, and AGIH a parabolic fection by a plane perpendicular to the former, and parallel to the fide VM of the cone; alfo let AFH be the common interfection of the two planes, or the axis of the parabola, and FG, HI ordinates perpendicular to it.

Then it will be, as AF : AH :: $FG^2 : HI^2$.

For, through the ordinates FG, HI draw the circular fections KGL, MIN, parallel to the bafe of the cone, having KL, MN for their diameters, to which FG, HI are ordinates, as well as to the axis of the parabola.

Then,

Then, by similar triangles, $AF : AH :: FL : HN$;
but, because of the parallels, $KF = MH$;
therefore - - - $AF : AH :: KF . FL : MH . HN$.
But, by the circle, $KF . FL = FG^2$, and $MH . HN = HI^2$;
Therefore - - $AF : AH :: FG^2 : HI^2$. Q. E. D.

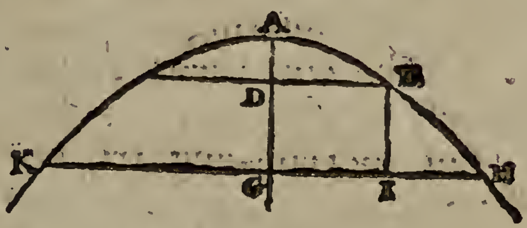
Corol. Hence the third proportional $\frac{FG^2}{AF}$ or $\frac{HI^2}{AH}$ is a constant quantity, and is equal to the parameter of the axis by defin. 16.

Or $AF : FG :: FG : P$ the parameter.
Or the rectangle $P . AF = FG^2$.

THEOREM II.

As the Parameter of the Axis :
Is to the Sum of any Two Ordinates ::
So is the Difference of those Ordinates :
To the Difference of their Abscisses :

That is,
$P : GH + DE :: GH - DE : DG$,
Or,
$P : KI :: IH : IE$.



FOR, by cor. theor. 1, $P . AG = GH^2$,
and - - $P . AD = DE^2$;
theref. by subtraction, $P . DG = GH^2 - DE^2$.
Or, - - $P . DG = KI . IH$,
therefore - - $P : KI :: IH : DG$ or EI. Q. E. D.

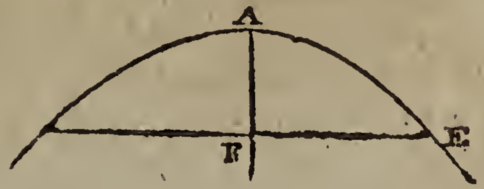
Corol. Hence, because $P . EI = KI . IH$,
and, by cor. theor. 1, $P . AG = GH^2$,
therefore - $AG : EI :: GH^2 : KI . IH$.

So that any diameter EI is as the rectangle of the segments KI, IH of the double ordinate KH.

THEOREM III.

The Distance from the Vertex to the Focus is equal to $\frac{1}{4}$ of the Parameter, or to Half the Ordinate at the Focus.

That is,
$AF = \frac{1}{2}FE = \frac{1}{4}P$,
where F is the focus.

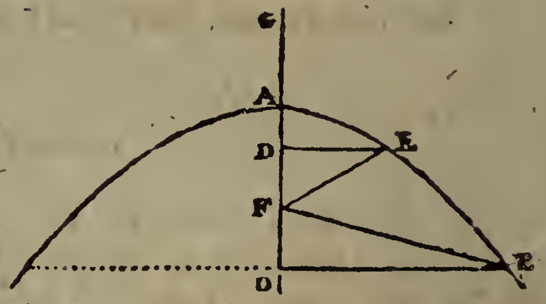


For, the general property is $AF : FE :: FE : P$.
But, by definition 17, - $FE = \frac{1}{2}P$;
therefore also - - $AF = \frac{1}{2}FE = \frac{1}{4}P$. Q. E. D.

THEOREM IV.

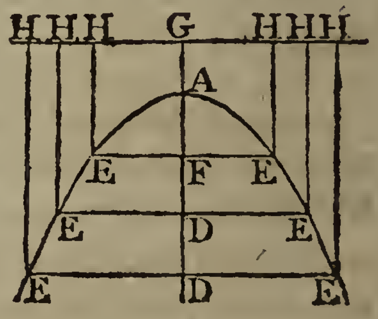
A Line drawn from the Focus to any Point in the Curve, is equal to the Sum of the Focal Distance and the Abscifs of the Ordinate to that Point.

That is,
$FE = FA + AD = GD$,
taking $AG = AF$.



FOR, since $FD = AD \backsim AF$,
theref. by squaring, $FD^2 = AF^2 - 2AF.AD + AD^2$,
But, by cor. theor. 1, $DE^2 = P.AD = 4AF.AD$;
theref. by addition, $FD^2 + DE^2 = AF^2 + 2AF.AD + AD^2$,
But, by right-ang. tri. $FD^2 + DE^2 = FE^2$;
therefore - $FE^2 = AF^2 + 2AF.AD + AD^2$,
and the root or side is $FE = AF + AD$,
or - - $FE = GD$, by taking $AG = AF$.
Q. E. D.

Corol. 1. If, through the point G, the line HG be drawn perpendicular to the axis, it is called the directrix of the parabola. The property of which, from this theorem, it appears, is this: That drawing any lines HE parallel to the axis, HE is always equal to FE the distance of the focus from the point E.



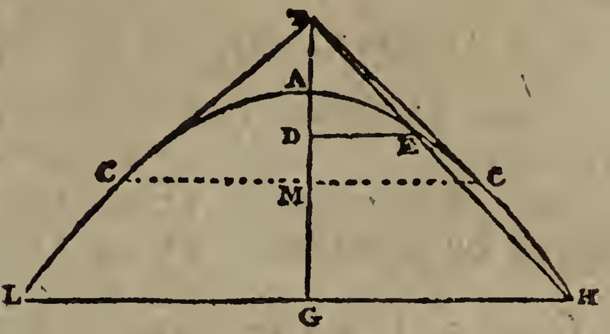
Corol. 2. Hence also the curve is easily described by points. Namely, in the axis produced take $AG = AF$ the focal distance, and draw a number of lines EE perpendicular to the axis AD; then with the distances GD, GD, GD, &c, as radii, and the centre F, draw arcs crossing the parallel ordinates in E, E, E, &c. Then draw the curve through all the points E, E, E.

THE-

THEOREM V.

If a Tangent be drawn to any Point of the Parabola, meeting the Axis produced; and if an Ordinate to the Axis be drawn from the Point of Contact; then the Abscis of that Ordinate will be equal to the External Part of the Axis.

That is,
if TC touch the curve
at the point C;
then is $AT = AM$.



FOR, from the point T, draw any line cutting the curve in the two points E, H: to which draw the ordinates DE, GH; also draw the ordinate MC to the point of contact C.

Then, by th. 1, $AD : AG :: DE^2 : GH^2$;
and, by sim. tri. $TD^2 : TG^2 :: DE^2 : GH^2$;
theref. by equality, $AD : AG :: TD^2 : TG^2$;
and, by division, $AD : DG :: TD^2 : TG^2 - TD^2$ or $DG.\overline{TD + TG}$,
or - $AD : TD :: TD : TD + TG$;
and, by division, $AD : AT :: TD : TG$,
and again by div. $AD : AT :: AT : AG$;
or - AT is a mean propor. between AD, AG.

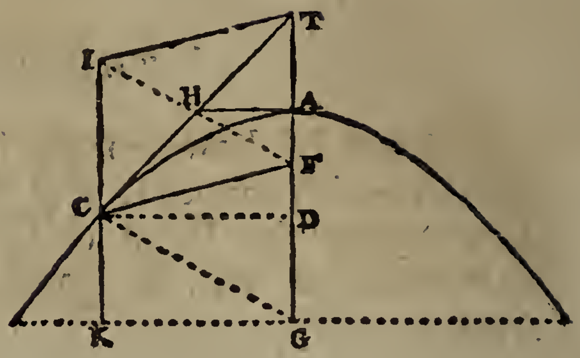
Now, if the line TH be supposed to revolve about the point T; then, as it recedes farther from the axis, the points E and H approach towards each other, the point E descending, and the point H ascending, till at last they meet in the point C, when the line becomes a tangent to the curve at C. And then the points D and G meet in the point M, and the ordinates DE, GH in the ordinate CM. Consequently AD, AG, becoming each equal to AM, their mean proportional AT will be equal to the abscis AM. That is, the external part of the axis, cut off by a tangent, is equal to the abscis of the ordinate to the point of contact. Q. E. D.

THEOREM VI.

If a tangent to the Curve meet the Axis produced; then the Line drawn from the Focus to the Point of Contact, will be equal to the Distance of the Focus from the Intersection of the Tangent and Axis.

That

That is,
FC = FT.



For, draw the ordinate DC to the point of contact C.

Then, by theor. 5, AT = AD;
therefore - FT = AF + AD.
But, by theor. 4, FC = AF + AD;
theref. by equality, FC = FT. Q. E. D.

Corol. 1. If CG be drawn perpendicular to the curve, or to the tangent, at C; then shall FG = FC = FT.

For, draw FH perpendicular to TC, which will also bisect TC, because FT = FC; and therefore, by the nature of the parallels, FH also bisects TG in F. And consequently FG = FT = FC.

So that F is the centre of a circle passing through T, C, G.

Corol. 2. The tangent at the vertex AH, is a mean proportional between AF and AD.

For, because FHT is a right angle,
therefore - AH is a mean between AF, AT,
or between AF, AD, because AD = AT,
Likewise, FH is a mean between FA, FT,
or between FA, FC.

Corol. 3. The tangent TC makes equal angles with FC and the axis FT.

For, because FT = FC,
therefore the ∠ FCT = ∠ FTC.
Also, the angle GCF = the angle GCK,
drawing ICK parallel to the axis AG.

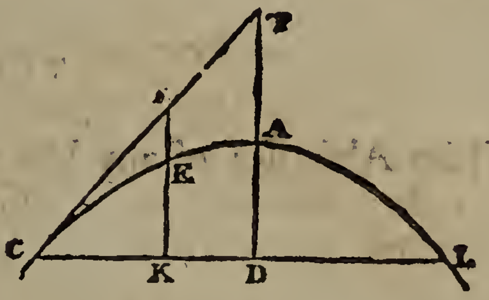
Corol. 4. And because the angle of incidence GCK is = the angle of reflection GCF; therefore a ray of light falling on the curve in the direction KC, will be reflected to the focus F. That is, all rays parallel to the axis, are reflected to the focus, or burning point.

THE

THEOREM VII.

If there be any Tangent, and a Double Ordinate drawn from the Point of Contact, and also any Line parallel to the Axis, limited by the Tangent and Double Ordinate: then shall the Curve divide that Line in the same Ratio, as the Line divides the Double Ordinate.

That is,
IE : EK :: CK : KL.

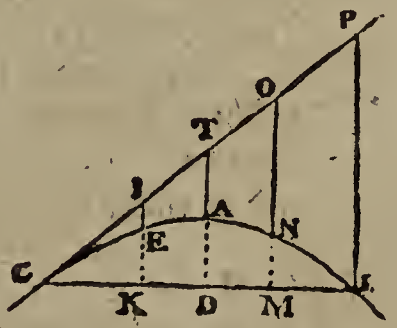


FOR, by sim. triangles,	CK : KI :: CD : DT or 2DA ;
but, by the def. the param.	P : CL :: CD : 2DA ;
therefore, by equality,	P : CK :: CL : KI.
But, by theor. 2, -	P : CK :: KL : KE ;
therefore, by equality,	CL : KL :: KI : KE ;
and, by division, -	CK : KL :: IE : EK. Q. E. D.

THEOREM VIII.

The same being supposed as in theor. 7 ; then shall the External Part of the Line between the Curve and Tangent, be proportional to the Square of the intercepted Part of the Tangent, or to the Square of the intercepted Part of the Double Ordinate.

That is, IE is as CI^2 or as CK^2.
and IE, TA, ON, PL, &c,
are as CI^2, CT^2, CO^2, CP^2, &c,
or as CK^2, CD^2, CM^2, CL^2, &c.



FOR, by theor. 7, IE : EK :: CK : KL,
or, by equality, IE : EK :: CK^2 : CK . KL.
But, by cor. th. 2, EK is as the rect. CK . KL,
therefore IE is as CK^2, or as CI^2. Q. E. D.

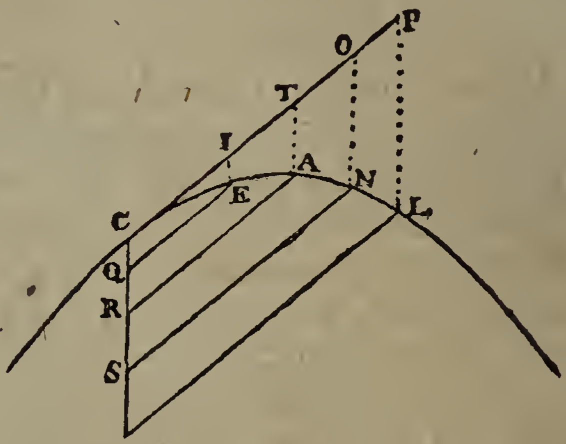
Corol

Corol. As this property is common to every poſition of the tangent, if the lines IE, TA, ON, &c, be appended to the points I, T, O, &c, and moveable about them, and of ſuch lengths as that their extremities E, A, N, &c, be in the curve of a parabola in ſome one poſition of the tangent; then making the tangent revolve about the point C, it appears that the extremities E, A, N, &c, will always form the curve of ſome parabola, in every poſition of the tangent.

THEOREM IX.

The Abſciſſes of any Diameter, are as the Squares of their Ordinates.

That is, CQ, CR, CS, &c, are as QE^2, RA^2, SN^2, &c.
Or - $CQ : CR :: QE^2 : RA^2$, &c.



FOR, draw the tangent CT, and the externals EI, AT, NO, &c, parallel to the axis, or to the diameter CS.

Then, becauſe the ordinates QE, RA, SN, &c, are parallel to the tangent CT, by the definition of them, therefore all the figures IQ, TR, OS, &c, are parallelograms, whoſe oppoſite ſides are equal,

namely,	IE,	TA,	ON,	&c,
are equal to	CQ,	CR,	CS,	&c.
Therefore, by theor. 8,	CQ,	CR,	CS,	&c,
are as -	CI^2,	CT^2,	CO^2,	&c,
or as their equals	QE^2,	RA^2,	SN^2,	&c,

Q. E. D.

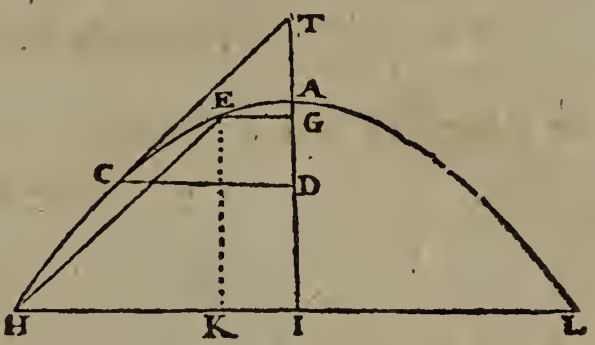
Corol. Here, like as in theor. 2, the difference of the abſciſſes is as the difference of the ſquares of their ordinates, or as the rectangles under the ſum and difference of the ordinates, the rectangle under the ſum and difference of the ordinates being equal to the rectangle under the difference of the abſciſſes and the parameter of that diameter, or a third proportional to any abſciſs and its ordinate.

THE-

THEOREM X.

If a Line be drawn parallel to any Tangent, and cut the Curve in two Points; then if two Ordinates be drawn to the Interfections, and a third to the Point of Contact, thefe three Ordinates will be in Arithmetical Progreffion, or the Sum of the Extremes will be equal to Double the Mean.

That is,
$EG + HI = 2CD$.

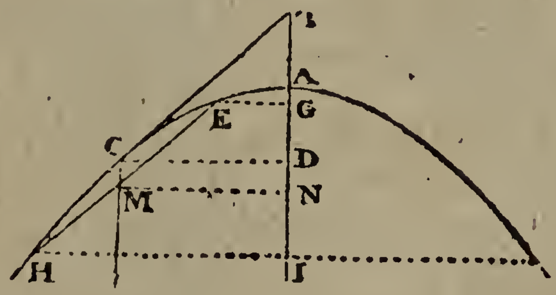


FOR, draw EK parallel to the axis, and produce HI to L.
Then, by fim. triangles, $EK : HK :: TD$ or $2AD : CD$;
but, by theor. 2, - $EK : HK :: KL : P$ the param.
theref. by equality, - $2AD : KL :: CD : P$.
But, by the defin. - $2AD : 2CD :: CD : P$;
theref. the 2d terms are equal, $KL = 2CD$,
that is, - - $EG + HI = 2CD$. Q. E. D.

THEOREM XI.

Any Diameter bifects all its Double Ordinates, or Lines parallel to the Tangent at its Vertex.

That is,
$ME = MH$.



FOR, to the axis AI draw the ordinates EG, CD, HI, and MN parallel to them, which is equal to CD.
Then, by theor. 10, $2MN$ or $2CD = EG + HI$,
therefore M is the middle of EH.
And, for the fame reafon, all its parallels are bifected.
Q. E. D.

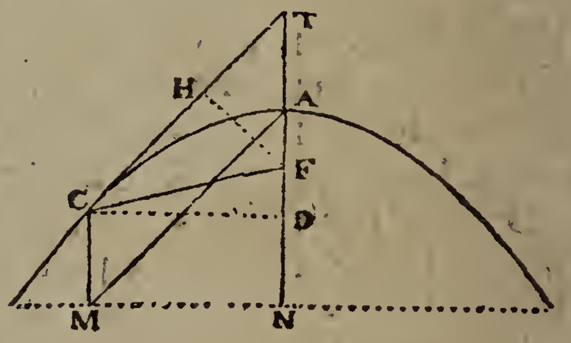
SCHOL.

Schol. Hence, as the abſciſſes of any diameter and their ordinates have the ſame relations as thoſe of the axis, namely, that the ordinates are biſected by the diameter, and their ſquares proportional to the abſciſſes; ſo all the other properties of the axis and its ordinates and abſciſſes, before demonſtrated, will likewiſe hold good for any diameter and its ordinates and abſciſſes. And alſo thoſe of the parameters, underſtanding the parameter of any diameter, as a third proportional to any abſciſs and its ordinate. Some of the moſt material of which are demonſtrated in the following theorems:

THEOREM XII.

The Parameter of any Diameter is equal to four Times the Line drawn from the Focus to the Vertex of that Diameter.

That is, $4FC = p$, the param. of the diam. CM.



For, draw the ordinate MA parallel to the tangent CT: as alſo CD, MN perpendicular to the axis AN, and FH perpendicular to the tangent CT.

Then the abſciſſes AD, CM or AT being equal, by theor. 5, the parameters will be as the ſquares of the ordinates CD, MA or CT, by the definition;

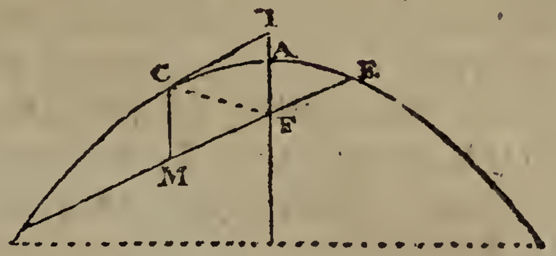
that is,	$P : p :: CD^2 : CT^2$,
But, by ſim. tri.	$FH : FT :: CD : CT$;
therefore -	$P : p :: FH^2 : FT^2$.
But, by cor. 2, th. 6,	$FH^2 = FA \cdot FT$;
therefore -	$P : p :: FA \cdot FT : FT^2$,
or, by equality,	$P : p :: FA : FT$ or FC.
But, by theor. 3,	$P = 4FA$,
and therefore -	$p = 4FT$ or $4FC$. Q. E. D.

Corol. Hence the parameter p of the diameter CM is equal to $4FA + 4AD$, or to $P + 4AD$, that is, the parameter of the axis added to $4AD$.

THEOREM XIII.

If an Ordinate to any Diameter, paſs through the Focus, it will be equal to Half its Parameter; and its Abſciſs equal to One Fourth of the ſame Parameter.

That is, CM $= \frac{1}{4}$p,
and ME $= \frac{1}{2}$p.

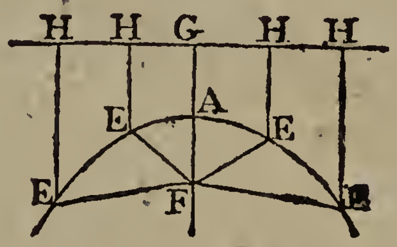


FOR, join FC, and draw the tangent CT.

By the parallels, CM = FT;
and, by theor. 6, FC = FT;
alſo, by theor. 12, FC = $\frac{1}{4}$p;
therefore - - CM = $\frac{1}{4}$p.
Again, by the defin. CM or $\frac{1}{4}$p : ME :: ME : p,
and conſequently - ME = $\frac{1}{2}$p = 2CM.

Corol. 1. Hence, of any diameter, the double ordinate which paſſeth through the focus, is equal to the parameter, or to quadruple its abſciſs.

Corol. 2. Hence, and from cor. 1 to theor. 4, and theor. 6 and 12, it appears, that if the directrix GH be drawn, and any lines HE, HE, parallel to the axis; then every parallel HE will be equal to EF, or $\frac{1}{4}$ of the parameter of the diameter to the point E.

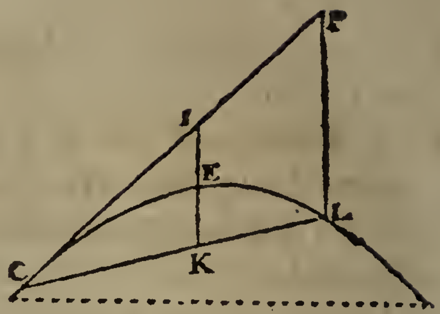


THEOREM XIV.

If there be a Tangent, and any Line drawn from the Point of Contact and meeting the Curve in ſome other Point, as alſo another Line parallel to the Axis, and limited by the Firſt Line and the Tangent: then ſhall the Curve divide this Second Line in the ſame Ratio, as the Second Line divides the Firſt Line.

That

That is,
IE : EK :: CK : KL.



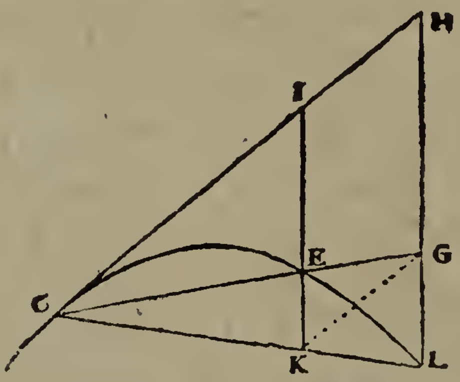
For, draw LP parallel to IK, or to the axis.

Then, by theor. 8,	IE : PL :: CI^2 : CP^2,
or, by ſim. tri.	IE : PL :: CK^2 : CL^2.
Alſo, by ſim. tri.	IK : PL :: CK : CL,
or	IK : PL :: CK^2 : CK.CL;
therefore by equality,	IE : IK :: CK.CL : CL^2;
or	IE : IK :: CK : CL;
and, by diviſion,	IE : EK :: CK : KL. Q. E. D.

Corol. When CK = KL, then IE = EK = $\frac{1}{2}$IK.

THEOREM XV.

If from any Point of the Curve there be drawn a Tangent, and alſo Two Right Lines to cut the Curve; and Diameters be drawn through the Points of Interſection E and L, meeting thoſe Two Right Lines in two other Points G and K: Then will the Line KG joining theſe laſt Two Points be parallel to the Tangent.

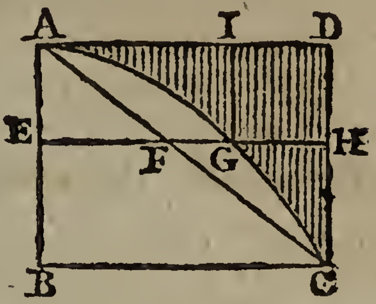


For, by theor. 14,	CK : KL :: EI : EK;
and by comp. -	CK : CL :: EI : KI;
and by the parellels	:: GH : LH;
But, by ſim. tri.	CK : CL :: KI : LH;
theref. by equal.	KI : LH :: GH : LH:
conſequently -	KI = GH,
and therefore -	KG is parallel and equal to IH. Q. E. D.

THE-

THEOREM XVI.

If a Rectangle be described about a Parabola, having the same Base and Altitude; and a diagonal Line be drawn from the Vertex to the Extremity of the Base of the Parabola, forming a right-angled Triangle, of the same Base and Altitude also; then any Line or Ordinate drawn across the three Figures, perpendicular to the Axis, will be cut in Continual Proportion by the Sides of those Figures.



That is,
EF : EG :: EG : EH,
Or, EF, EG, EH, are in continued proportion.

For, by theor. 1, AB : AE :: BC^2 : EG^2,
and, by sim. tri. - AB : AE :: BC : EF,
theref. of equality, EF : BC :: EG^2 : BC^2,
that is, - EF : EH :: EG^2 : EH^2,
theref. by Geom. th. 78, EF, EG, EH are proportionals,
or - - EF : EG :: EG : EH. Q. E. D.

THEOREM XVII.

The Area or Space of a Parabola, is equal to Two-Thirds of its Circumscribing Parallelogram.

That is, the space ABCGA $= \frac{2}{3}$ABCD;
or, the space ADCGA $= \frac{1}{3}$ABCD.

FOR, conceive the space ADCGA to be composed of, or divided into, indefinitively small parts, by lines parallel to DC or AB, such as IG, which divide AD into like small and equal parts, the number or sum of which is expressed by the line AD. Then,

by the parabola, BC^2 : EG^2 :: AB : AE,
that is, - AD^2 : AI^2 :: DC : IG.

Hence it follows, that any one of these narrow parts, as IG, is $= \frac{DC}{AD^2} \times AI^2$, whence, AD and DC being given or constant quantities, it appears that the said parts IG, &c, are proportional to AI^2, &c, or proportional to a series of square numbers,

numbers, whoſe roots are in arithmetical progreſſion, and the area ADCGA equal to $\frac{DC}{AD^2}$ drawn into the ſum of ſuch a ſeries of arithmeticals, the number of which is expreſſed by AD.

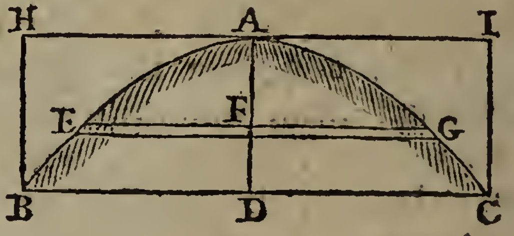
Now, by the remark at pag. 233, vol. i, the ſum of the ſquares of ſuch a ſeries of arithmeticals, is expreſſed by $\frac{1}{6}n . \overline{n+1} . \overline{2n+1}$, where n denotes the number of them. In the preſent caſe, n repreſents an infinite number, and then the ſubfactors $n+1$, $2n+1$, become only n and $2n$, omitting the 1 as inconſiderable in reſpect of the infinite number n: hence the expreſſion above becomes barely $\frac{1}{6}n . n . 2n = \frac{1}{3}n^3$.

To apply this to the caſe above: n will denote AD or BC; and the ſum of all the AI^2's becomes $\frac{1}{3}AD^3$ or $\frac{1}{3}BC^3$; conſequently the ſum of all the $\frac{DC}{AD^2} \times AI^2$'s, is $\frac{DC}{AD^2} \times \frac{1}{3}AD^3 = \frac{1}{3}AD . DC = \frac{1}{3}BD$, which is the area of the exterior part ADCGA. That is, the ſaid exterior part ADCGA, is $\frac{1}{3}$ of the parallelogram ABCD; and conſequently the interior part ABCGA is $\frac{2}{3}$ of the ſame parallelogram. Q. E. D.

Corol. The part AFCGA, incloſed between the curve and the right line AFC, is $\frac{1}{6}$ of the ſame parallelogram.

THEOREM XVIII.

The Solid Content of a Paraboloid (or Solid generated by the Rotation of a Parabola about its Axis), is Half its Circumſcribing Cylinder.



LET ABC be a paraboloid, generated by the rotation of the parabola AC about its axis AD. Suppoſe the axis AD be divided into an infinite number of equal parts, through which let circular planes paſs, as EFG, all thoſe circles making up the whole ſolid paraboloid.

Then, if $c =$ the number 3·1416, then $2c \times FG$ is the circumference of the circle EFG whoſe radius is FG; therefore $c \times FG^2$ is the area of that circle.

But, by cor. theor. 1, Parabola, $p \times AF = FG^2$, where p denotes the parameter of the parabola; conſequently $pc \times AF$ will

will alſo expreſs the ſame circular ſection EG, and therefore $pc \times$ the ſum of all the AF's will be the ſum of all thoſe circular ſections, or the whole content of the ſolid paraboloid.

But all the AF's form an arithmetical progreſſion, beginning at 0 or nothing, and having the greateſt term and the ſum of all the terms each expreſſed by the whole axis AD. And ſince the ſum of all the terms of ſuch a progreſſion, is equal to $\frac{1}{2}AD \times AD$ or $\frac{1}{2}AD^2$, half the product of the greateſt term and the number of terms; therefore $\frac{1}{2}AD^2$ is equal to the ſum of all the AF's, and conſequently, $pc \times \frac{1}{2}AD^2$, or $\frac{1}{2}c \times p \times AD^2$, is the ſum of all the circular ſections, or the content of the paraboloid.

But, by the parabola, $p : DC :: DC : AD$ or $p = \frac{DC^2}{AD}$; conſequently $\frac{1}{2}c \times p \times AD^2$ becomes $\frac{1}{2}c \times AD \times DC^2$ for the ſolid content of the paraboloid. But $c \times AD \times DC^2$ is equal to the cylinder BCIH; conſequently the paraboloid is the half of its circumſcribing cylinder. Q. E. D.

THEOREM XIX.

The Solidity of the Fruſtum BEGC of the Paraboloid, is equal to a Cylinder whoſe Height is DF, and its Baſe Half the Sum of the two Circular Baſes EG, BC.

FOR, by laſt theor. $\frac{1}{2}pc \times AD^2 =$ the ſolid ABC,
and - - $\frac{1}{2}pc \times AF^2 =$ the ſolid AEG,
theref. the dif. $\frac{1}{2}pc \times (AD^2 - AF^2) =$ the fruſt. BEGC.
But $AD^2 - AF^2 = DF \times (AD + AF)$,
theref. $\frac{1}{2}pc \times DF \times (AD + AF) =$ the fruſt. BEGC.
But, by the parab. $p \times AD = DC^2$, and $p \times AF = FG^2$;
theref. $\frac{1}{2}c \times DF \times (DC^2 + FG^2) =$ the fruſt. BEGC.
Q. E. D.

Of MOTION, FORCES, &c.

DEFINITIONS.

Art. 1. BODY is the mafs, or quantity of matter, in any material fubftance; and it is always proportional to its weight or gravity, whatever its figure may be.

2. Body is either Hard, Soft, or Elaftic. A Hard Body is that whofe parts do not yield to any ftroke or percuffion, but retains its figure unaltered. A Soft Body is that whofe parts yield to any ftroke or impreffion, without reftoring themfelves again; the figure of the body remaining altered. And an Elaftic Body is that whofe parts yield to any ftroke, but which prefently reftore themfelves again, and the body regains the fame figure as before the ftroke.

We know of no bodies that are abfolutely, or perfectly, either hard, foft, or elaftic; but all partaking thefe properties, more or lefs, in fome intermediate degree.

3. Bodies are alfo either Solid or Fluid. A Solid Body, is that whofe parts are not eafily moved amongft one another, and which retains any figure you give it. But a Fluid Body is that whofe parts yield to the flighteft impreffion, being eafily moved amongft one another; and its furface, when left to itfelf, is always obferved to fettle in a fmooth plane at the top.

4. Denfity is the proportional weight or quantity of matter in any body.

So, in two fpheres, or cubes, &c, of equal fize or magnitude; if the one weigh only one pound, but the other two pounds; then the denfity of the latter is double the denfity of the former; if it weigh three pounds, its denfity is triple, and fo on.

5. Motion is a continual and fucceffive change of place.—If the body move equally, or pafs over equal fpaces in equal times, it is called Equable or Uniform Motion. But if it increafe or decreafe, it is Variable Motion; and it is called Accelerated Motion in the former cafe, and Retarded Motion in the latter.—Alfo, when the body moved is confidered with

refpect

refpect to fome other body at reft, it is faid to be Abfolute Motion. But when compared with others in motion, it is called Relative Motion.

6. Velocity, or Celerity, is an affection of motion, by which a body paffes over a certain fpace in a certain time. Thus, if a body in motion pafs uniformly over 40 feet in 4 feconds of time, it is faid to move with the velocity of 10 feet per fecond; and fo on.

7. Momentum, or Quantity of Motion, is the power or force incident to moving bodies, by which they continually tend from their prefent places, or with which they ftrike any obftacle that oppofes their motion.

8. Force is a power exerted on a body to move it, or to ftop it. If the force act conftantly, or inceffantly, it is a Permanent Force: like preffure or the force of gravity. But if it act inftantaneoufly, or but for an imperceptibly fmall time, it is called Impulfe, or Percuffion: like the fmart blow of a hammer.

9. Forces are alfo diftinguifhed into Motive, and Accelerative or Retarding. A Motive or Moving Force, is the power of an agent to produce motion; and it is equal or proportional to the momentum it will generate in any body, when acting, either by percuffion, or for a certain time as a permanent force.

10. Accelerative, or Retardive Force, is commonly underftood to be that which affects the velocity only; or it is that by which the velocity is accelerated or retarded; and it is equal or proportional to the motive force directly and to the mafs or body moved inverfely.—So, if a body of 2 pounds weight, be acted on by a motive force of 40; then the accelerating force is 20. But if the fame force of 40 act on another body of 4 pounds weight; then the accelerating force in this latter cafe is only 10; and fo is but half the former.

11. Gravity, or Weight, is that force by which a body endeavours to fall downwards. It is called Abfolute Gravity, when the body is in empty fpace; and Relative Gravity, when immerfed in a fluid.

12. Specific Gravity is the proportion of the weights of different bodies of equal magnitude; and fo is proportional to the denfity of the body.

AXIOMS.

13. EVERY body naturally endeavours to continue in its present ſtate, whether it be at reſt, or moving uniformly in a right line.

14. The Change or Alteration of Motion, by any external force, is always proportional to that force, and in the direction of the right line in which it acts.

15. Action and Re-action, between any two bodies, are equal and contrary. That is, by Action and Re-action, equal changes of motion are produced in bodies acting on each other; and theſe changes are directed towards oppoſite or contrary parts.

GENERAL LAWS OF MOTION, &c.

PROPOSITION I.

16. *The Quantity of Matter, in all Bodies, is in the Compound Ratio of their Magnitudes and Denſities.*

THAT is, b is as md; where b denotes the body or quantity of matter, m its magnitude, and d its denſity.

For, by art. 4, in bodies of equal magnitude, the maſs or quantity of matter is as the denſity. But, the denſities remaining, the maſs is as the magnitude; that is, a double magnitude contains a double quantity of matter, a triple magnitude a triple quantity, and ſo on. Therefore the maſs is in the compound ratio of the magnitude and denſity.

17. *Corol.* 1. In ſimilar bodies, the maſſes are as the denſities and cubes of the diameters, or of any like linear dimenſions.—For the magnitudes of bodies are as the cubes of the diameters, &c.

18. *Corol.* 2. The maſſes are as the magnitudes and ſpecific gravities.—For, by art. 4 and 12, the denſities of bodies are as the ſpecific gravities.

19. *Scho-*

19. *Scholium.* Hence, if b denote any body, or the quantity of matter in it, m its magnitude, d its denſity, g its ſpecific gravity, and a its diameter or other dimenſion; then, $\propto$ (pronounced or named *as*) being the mark for general proportion, from this propoſition and its corollaries we have theſe general proportions:

$$b \propto md \propto mg \propto a^3 d,$$
$$m \propto \frac{b}{d} \propto \frac{b}{g} \propto a^3,$$
$$d \propto \frac{b}{m} \propto g \propto \frac{mg}{a^3},$$
$$a^3 \propto \frac{b}{d} \propto m \propto \frac{mg}{d}.$$

PROPOSITION II.

20. *The Momentum, or Quantity of Motion, generated by a Single Impulſe, or any Momentary Force, is as the Generating Force.*

THAT is, m is as f; where m denotes the momentum, and f the force.

For every effect is proportional to its adequate cauſe. So that a double force will impreſs a double quantity of motion; a triple force, a triple motion; and ſo on. That is, the motion impreſſed, is as the motive force which produces it.

PROPOSITION III.

21. *The Momenta, or Quantities of Motion, in Moving Bodies, are in the Compound Ratio of the Maſſes and Velocities.*

That is, m is as bv.

FOR, the motion of any body being made up of the motions of all its parts, if the velocities be equal, the momenta will be as the maſſes; for a double maſs will ſtrike with a double force; a triple maſs, with a triple force; and ſo on. Again, when the maſs is the ſame, it will require a double force to move it with a double velocity, a triple force with a triple velocity, and ſo on; that is, the motive force is as the velocity; but the momentum impreſſed, is as the force which produces it, by prop. 2; and therefore the momentum is as the velocity when the maſs is the ſame. But the momentum was found to be as the maſs when the velocity is the ſame.

 Conſe-

Consequently, when neither are the same, the momentum is in the compound ratio of both the mass and velocity.

PROPOSITION IV.

22. *In Uniform Motions, the Spaces described are in the Compound Ratio of the Velocities and the Times of their Description.*

That is, s is as tv.

For, by the nature of uniform motion, the greater the velocity, the greater is the space described in any one and the same time; that is, the space is as the velocity, when the times are equal. And when the velocity is the same, the space will be as the time; that is, in a double time a double space will be described; in a triple time, a triple space; and so on. Therefore universally, the space is in the compound ratio of the velocity, and the time of description.

23. *Corol.* 1. In uniform motions, the time is as the space directly, and velocity reciprocally; or as the space divided by the velocity. And when the velocity is the same, the time is as the space. But when the space is the same, the time is reciprocally as the velocity.

24. *Corol.* 2. The velocity is as the space directly and the time reciprocally; or as the space divided by the time. And when the time is the same, the velocity is as the space. But when the space is the same, the velocity is reciprocally as the time.

Scholium.

25. In uniform motions generated by momentary impulse,
let b = any body or quantity of matter to be moved,
f = force of impulse acting on the body b,
v = the uniform velocity generated in b,
m = the momentum generated in b,
s = the space described by the body b,
t = the time of describing the space s with the veloc. v.

Then from the last three propositions and corollaries, we have these three general proportions, namely, $f \propto m$, $m \propto bv$, and $s \propto tv$; from which is derived the following table of the general relations of those six quantities, in uniform motions, and impulsive or percussive forces:

$f \propto m$

$$f \propto m \propto bv \propto \frac{bs}{t}.$$

$$m \propto f \propto bv \propto \frac{bs}{t}.$$

$$b \propto \frac{f}{v} \propto \frac{m}{v} \propto \frac{ft}{s} \propto \frac{mt}{s}.$$

$$s \propto tv \propto \frac{tf}{b} \propto \frac{tm}{b}.$$

$$v \propto \frac{s}{t} \propto \frac{f}{b} \propto \frac{m}{b}.$$

$$t \propto \frac{s}{v} \propto \frac{bs}{f} \propto \frac{bs}{m}.$$

By means of which, may be reſolved all queſtions relating to uniform motions, and the effects of momentary or impulſive forces.

PROPOSITION V.

26. *The Momentum generated by a Conſtant and Uniform Force, acting for any Time, is in the Compound Ratio of the Force and Time of Acting.*

That is, m is as ft.

FOR, ſuppoſing the time divided into very ſmall parts, by prop. 2, the momentum in each particle of time is the ſame, and therefore the whole momentum will be as the whole time, or ſum of all the ſmall parts. But, by the ſame prop. the momentum for each ſmall time, is alſo as the motive force. Conſequently the whole momentum generated, is in the compound ratio of the force and time of acting.

27. *Corol.* 1. The motion, or momentum, loſt or deſtroyed in any time, is alſo in the compound ratio of the force and time. For whatever momentum any force generates in a given time; the ſame momentum will an equal force deſtroy in the ſame or equal time; acting in a contrary direction.

And the ſame is true of the increaſe or decreaſe of motion, by forces that conſpire with, or oppoſe, the motion of bodies.

28. *Corol.* 2. The velocity generated, or deſtroyed, in any time, is directly as the force and time, and reciprocally as the body or maſs of matter.—For, by this and the 3d prop. the compound ratio of the body and velocity, is as that of the force and time; and therefore the velocity is as the force and time divided by the body. And if the body and force be given, or conſtant, the velocity will be as the time.

PRO-

PROPOSITION VI.

29. *The Spaces passed over by Bodies, urged by any Constant and Uniform Forces, acting during any Times, are in the Compound Ratio of the Forces and Squares of the Times directly, and the Body or Mass reciprocally.*

Or, the Spaces are as the Squares of the Times, when the Force and Body are given.

THAT is, s is as $\frac{ft^2}{b}$, or as t^2 when f and b are given. For, let v denote the velocity acquired at the end of any time t, by any given body b, when it has passed over the space s. Then, because the velocity is as the time, by the last corol. therefore $\frac{1}{2}v$ is the velocity at $\frac{1}{2}t$, or at the middle point of the time; and as the increase of velocity is uniform, the same space s will be described in the same time t, by the velocity $\frac{1}{2}v$ uniformly continued from beginning to end. But, in uniform motions, the space is in the compound ratio of the time and velocity; therefore s is as $\frac{1}{2}tv$, or indeed $s = \frac{1}{2}tv$. But, by the last corol. the velocity v is as $\frac{ft}{b}$, or as the force and time directly, and as the body reciprocally. Therefore s, or $\frac{1}{2}tv$, is as $\frac{ft^2}{b}$, that is, the space is as the force and square of the time directly, and as the body reciprocally. Or s is as t^2, the square of the time only, when b and f are given.

30. *Corol.* 1. The space s is also as tv, or in the compound ratio of the time and velocity; b and f being given. For, $s = \frac{1}{2}tv$ is the space actually described. But tv is the space which might be described in the same time t, with the last velocity v, if it were uniformly continued for the same or an equal time. Therefore the space s, or $\frac{1}{2}tv$, which is actually described, is just half the space, which would be described with the last or greatest velocity, uniformly continued for an equal time t.

31. *Corol.* 2. The space s is also as v^2, the square of the velocity; because the velocity v is as the time t.

Scholium.

32. Propositions 3, 4, 5, 6, give theorems for resolving all questions relating to motions uniformly accelerated. Thus, put $b =$ any body or quantity of matter,

$f =$ the force constantly acting on it,

$t =$ the time of its acting,
$v =$ the velocity generating in the time t,
$s =$ the ſpace deſcribed in that time,
$m =$ the momentum at the end of the time.

Then, from theſe fundamental relations, $m \propto bv$, $m \propto ft$, $s \propto tv$, and $v \propto \frac{ft}{b}$, we obtain the following table of the general relations of uniformly accelerated motions:

$$m \propto bv \propto ft \propto \frac{bs}{t} \propto \frac{fs}{v} \propto \frac{ft^2v}{s} \propto \sqrt{bfs} \propto \sqrt{bftv}.$$
$$b \propto \frac{m}{v} \propto \frac{ft}{v} \propto \frac{mt}{s} \propto \frac{ft^2}{s} \propto \frac{f^2t^3}{ms} \propto \frac{m^2}{fs} \propto \frac{m^2}{ftv} \propto \frac{fs}{v^2}.$$
$$f \propto \frac{m}{t} \propto \frac{bv}{t} \propto \frac{mv}{s} \propto \frac{ms}{t^2v} \propto \frac{m^2}{bs} \propto \frac{m^2}{btv} \propto \frac{bv^2}{s} \propto \frac{bs}{t^2}.$$
$$v \propto \frac{s}{t} \propto \frac{ft}{b} \propto \frac{m}{b} \propto \frac{ms}{ft^2} \propto \frac{fs}{m} \propto \frac{m^2}{bft} \propto \sqrt{\frac{fs}{b}} \propto \frac{f^2st}{m^2}.$$
$$s \propto tv \propto \frac{ft^2}{b} \propto \frac{mt}{b} \propto \frac{ft^2v}{m} \propto \frac{mv}{f} \propto \frac{m^2}{bf} \propto \frac{bv^2}{f} \propto \frac{m^2v}{f^2t}.$$
$$t \propto \frac{s}{v} \propto \frac{m}{f} \propto \frac{bv}{f} \propto \frac{bs}{m} \propto \sqrt{\frac{bs}{f}} \propto \sqrt{\frac{ms}{fv}} \propto \frac{m^2}{bfv} \text{ \&c.}$$

33. And from theſe proportions thoſe quantities are to be left out which are given, or which are proportional to each other. Thus, if the body or quantity of matter be always the ſame, then the ſpace deſcribed is as the force and ſquare of the time. And if the body be proportional to the force, as all bodies are in reſpect to their gravity; then the ſpace deſcribed is as the ſquare of the time, or ſquare of the velocity; and in this caſe, if F be put $= \frac{f}{b}$, the accelerating force; then will

$$s \propto tv \propto \text{F}t^2 \propto \frac{v^2}{\text{F}}.$$
$$v \propto \frac{s}{t} \propto \text{F}t \propto \sqrt{\text{F}s}.$$
$$t \propto \frac{s}{v} \propto \frac{v}{\text{F}} \propto \sqrt{\frac{s}{\text{F}}}.$$

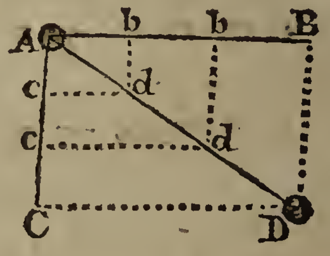
THE

THE COMPOSITION AND RESOLUTION OF FORCES.

34. COMPOSITION of FORCES, is the uniting of two or more forces into one, which ſhall have the ſame effect; or the finding of one force that ſhall be equal to ſeveral others taken together, in any different directions. And the Reſolution of Forces, is the finding of two or more forces which, acting in any different directions, ſhall have the ſame effect as any given ſingle force.

PROPOSITION VII.

35. *If a Body at* A *be urged in the Directions* AB *and* AC, *by any two Similar Forces, ſuch that they would ſeparately cauſe the Body to paſs over the Spaces* AB, AC *in an equal time; then if both Forces act together, they will cauſe the Body to move, in the ſame time, through* AD *the Diagonal of the Parallelogram* ABCD.



DRAW cd parallel to AB, and bd parallel to AC. And while the body is carried over Ab or cd by the force in that direction, let it be carried over bd by the force in that direction; by which means it will be found at d. Now, if the forces be impulſive or momentary, the motions will be uniform, and the ſpaces deſcribed will be as the times of deſcription;

theref. Ab or cd : AB or CD : : time in Ab : time in AB,
and cd or Ac : BD or AC : : time in Ac : time in AC;

but the time in Ab = time in Ac, and time in AB = time in AC, therefore Ab : bd : : AB : BD by equality.

And as this is always the caſe in every point d, d, &c, therefore the path of the body is the ſtraight line AdD, or the diagonal of the parallelogram.

But if the ſimilar forces, by means of which the body is moved in the directions AB, AC, be uniformly accelerating ones, then the ſpaces will be as the ſquares of the times; in which caſe, call the time in bd or cd t, and the time in AB or AC, T; then

it will be Ab or cd : AB or CD : : t^2 : T^2,
and - bd or AC : BD or AC : : t^2 : T^2,
theref. by equality, Ab : bd : : AB : BD;

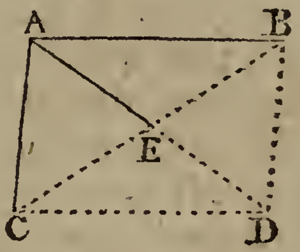
and ſo the body is always found in the diagonal, as before.

36. *Corol.*

36. *Corol.* 1. If the forces be not ſimilar, by which the body is urged in the directions AB, AC, it will move in ſome curve line, depending on the nature of the forces.

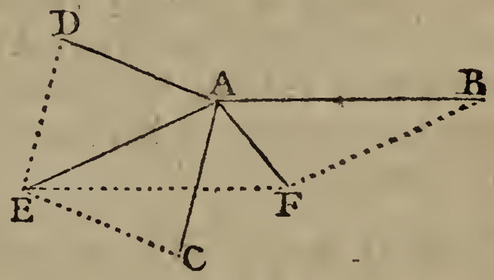
37. *Corol.* 2. Hence it appears, that the body moves over the diagonal AD, by the compound motion, in the very ſame time that it would move over the ſide AB, by the ſingle force impreſſed in that direction, or that it would move over the ſide AC by the force impreſſed in that direction.

38. *Corol.* 3. The forces in the directions AB, AC, AD, are reſpectively proportional to the lines AB, AC, AD, and in theſe directions.

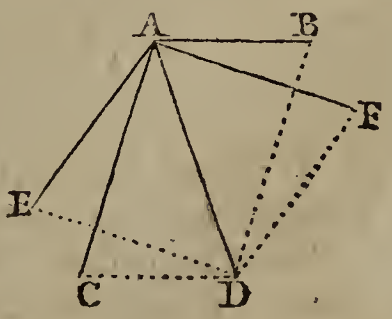


39. *Corol.* 4. The two oblique forces AB, AC, are equivalent to the ſingle direct force AD, which may be compounded of theſe two, by drawing the diagonal of the parallelogram. Or they are equivalent to the double of AE drawn to the middle of the line BC.

And thus any force may be compounded of two or more other forces; which is the meaning of the expreſſion, *compoſition of forces.*

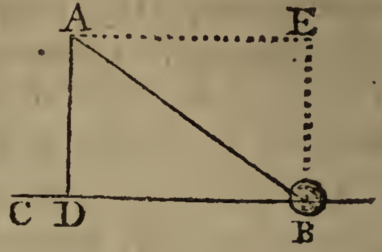


40. *Exam.* Suppoſe it were required to compound the three forces AB, AC, AD; or to find the direction and quantity of one ſingle force, which ſhall be equivalent to, and have the ſame effect as if a body at A were acted on by three forces in the directions AB, AC, AD, and proportional to theſe three lines. Firſt reduce the two AC, AD to one AE, by completing the parallelogram ADEC. Then reduce the two AE, AB to one AF, by the parallelogram AEFB. So ſhall the ſingle force AF be the direction, and as the quantity, which ſhall of itſelf produce the ſame effect, as if all the three AB, AC, AD acted together.



41. *Corol.* 5. Any ſingle direct force AD, may be reſolved into two oblique forces, whoſe quantities and directions are AB, AC, having the ſame effect, by deſcribing any parallelogram whoſe diagonal may be AD: and this is called the reſolution of forces. So the force AD may be reſolved into the two AB, AC, by the parallelogram ABDC;

ABDC; or into the two AE, AF, by the parallelogram AEDF; and so on, for any other two. And each of these may be resolved again into as many others as we please.



42. *Corol.* 6. Hence may be found the effect of any given force, in any other direction, besides that of the line in which it acts; as of the force AB in any other given direction CB. For draw AD perpendicular to CB; then shall DB be the effect of the force AB in the direction CB. For, the given force AB is equivalent to the two AD, DB, or AE; of which the former AD or EB, being perpendicular, does not alter the velocity in the direction CB; and therefore DB is the whole effect of AB in the direction CB. That is, a direct force expressed by the line DB acting in the direction DB, will produce the same effect or motion in a body B, in that direction, as the oblique force expressed by, and acting in, the direction AB, produces in the same direction CB. And hence a direct force DB, is to an oblique force AB, as AB to DB, or as radius to the cosine of the angle ABD of inclination of those forces. For the same reason, the force or effect in the direction AB, is to the force or effect in the direction AD or EB, as AB to AD; or as radius to sine of the same angle ABD, or cosine of the angle DAB of those directions.

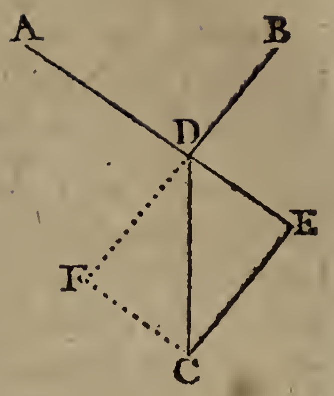
43. *Corol.* 7. Hence also, if the two given forces, to be compounded, act in the same line, either both the same way, or the one directly opposite to the other; then their joint or compounded force will act in the same line also, and will be equal to the sum of the two when they act the same way, or to the difference of them when they act in opposite directions; and the compound force, whether it be the sum or difference, will always act in the direction of the greater of the two.

PROPOSITION VIII.

44. *If Three Forces* A, B, C, *acting all together, in the same Plane, keep one another in Equilibrio; they will be Proportional to the Three Sides* DE, EC, CD, *of a Triangle, which are drawn Parallel to the Directions of the Forces* AD, DB, CD.

Produce AD, BD, and draw CF, CE parallel to them.

them. Then the force in CD is equivalent to the two AD, BD, by the supposition; but the force CD is also equivalent to the two ED and CE or FD; therefore, if CD represent the force C, then ED will represent its opposite force A, and CE or FD its opposite force B. Consequently the three forces A, B, C, are proportional to DE, CE, CD, the three lines parallel to the directions in which they act.

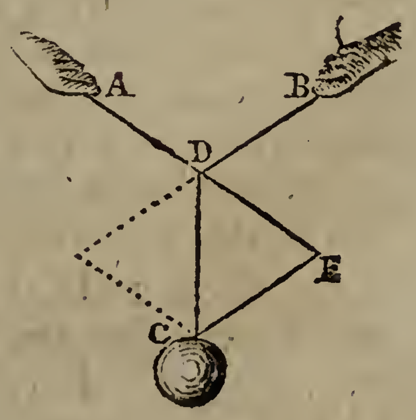


45. *Corol.* 1. Because the three sides CD, CE, DE, are proportional to the sines of their opposite angles E, D, C; therefore the three forces, when in equilibrio, are proportional to the sines of the angles of the triangle made of their lines of direction; namely, each force proportional to the sine of the angle made by the directions of the other two.

46. *Corol.* 2. The three forces, acting against, and keeping one another in equilibrio, are also proportional to the sides of any other triangle made by drawing lines either perpendicular to the directions of the forces, or forming any given angle with those directions. For such a triangle is always similar to the former, which is made by drawing lines parallel to the directions; and therefore their sides are in the same proportion to one another.

47. *Corol.* 3. If any number of forces be kept in equilibrio by their actions against one another; they may be all reduced to two equal and opposite ones.—For, by cor. 4, prop. 7, any two of the forces may be reduced to one force acting in the same plane; then this last force and another may likewise be reduced to another force acting in their plane: and so on, till at last they be all reduced to the action of only two opposite forces; which will be equal, as well as opposite, because the whole are in equilibrio by the supposition.

48. *Corol.* 4. If one of the forces, as C, be a weight, which is sustained by two strings drawing in the directions DA, DB: then the force or tension of the string AD, is to the weight C, or tension of the string DC, as DE to DC; and the force or tension of the other string BD, is to the weight C, or tension of CD, as CE to CD.



49. *Corol.*

49. *Corol.* 5. If three forces be in equilibrio by their mutual actions; the line of direction of each force, as DC, passes through the opposite angle C of the parallelogram formed by the directions of the other two forces.

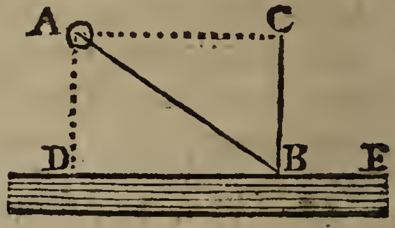
50. *Remark.* These properties, in this proposition and its corollaries, hold true of all similar forces whatever, whether they be instantaneous or continual, or whether they act by percussion, drawing, pushing, pressing, or weighing; and are of the utmost importance in mechanics and the doctrine of forces.

On the COLLISION of BODIES.

PROPOSITION IX.

51. *If a Body impinge or act Obliquely on a Plane Surface; the Force or Energy of the Stroke, or Action, is as the Sine of the Angle of Incidence.*

Or, the Force on the Surface is to the same if it had acted Perpendicularly, as the Sine of Incidence is to Radius.



Let AB express the direction and the absolute quantity of the oblique force on the plane DE; or let a given body A, moving with a certain velocity, impinge on the plane at B; then its force will be to the action on the plane, as radius to the sine of the angle ABD, or as AB to BC, drawing BC perpendicular, and AC parallel to DE.

For, by prop. 7, the force AB is equivalent to the two forces AC, CB; of which the former AC does not act on the plane, because it is parallel to it. The plane is therefore only acted on by the direct force CB, which is to AB, as the sine of the angle BAC, or ABD, to radius.

52. *Corol.* 1. If a body act on another, in any direction, and by any kind of force, the action of that force on the second body, is made only in a direction perpendicular to the surface on which it acts.

For the force in AB acts on DE only by the force CB, and in that direction.

53. *Corol.* 2. If the plane DE be not absolutely fixed, it will move, after the stroke, in the direction perpendicular to its surface. For it is in that direction that the force is exerted.

PRO-

PROPOSITION X.

54. *If one Body* A, *strike another Body* B, *which is either at Rest or moving towards the Body* A, *or moving from it, but with a less Velocity than that of* A; *then the Momenta, or Quantities of Motion, of the two Bodies, estimated in any one Direction, will be the very same after the Stroke that they were before it.*

FOR, because action and re-action are always equal, and in contrary directions, whatever momentum the one body gains one way by the stroke, the other must just lose as much in that same direction; and therefore the quantity of motion in that direction, resulting from the motions of both the bodies, remains still the same as it was before the stroke.

55. Thus, if A with a momentum of 10, strike B at rest, and communicate to it a momentum of 4, in the direction AB. Then A will have only a momentum of 6 in that direction; which, together with the momentum of B, viz. 4, make up still the same momentum between them as before, namely 10.

56. If B were in motion before the stroke, with a momentum of 5, in the same direction, and receive from A an additional momentum of 2. Then the motion of A after the stroke will be 8, and that of B, 7; which between them make 15, the same as 10 and 5, the motions before the stroke.

57. Lastly, if the bodies move in opposite directions, and meet one another, namely, A with a motion of 10, and B, of 5; and A communicate to B a motion of 6 in the direction AB of its motion. Then, before the stroke, the whole motion from both, in the direction of AB, is 10 — 5 or 5. But, after the stroke, the motion of A is 4 in the direction AB, and the motion of B is 6 — 5 or 1 in the same direction AB; therefore the sum 4 + 1, or 5, is still the same motion from both, as it was before.

PROPOSITION XI.

58. *The Motion of Bodies included in a Given Space, is the same, with regard to each other, whether that Space be at Rest, or move uniformly in a Right Line.*

FOR, if any force be equally impressed both on the body and the line in which it moves, this will cause no change in the

the motion of the body along the right line. For the ſame reaſon, the motions of all the other bodies, in their ſeveral directions, will ſtill remain the ſame. Conſequently their motions among themſelves will continue the ſame, whether the including ſpace be at reſt, or be moved uniformly forward. And therefore their mutual actions on one another, muſt alſo remain the ſame in both caſes.

PROPOSITION XII.

59. *If a Hard and Fixed Plane be ſtruck by either a Soft or a Hard Unelaſtic Body, the Body will adhere to it. But if the Plane be ſtruck by a Perfectly Elaſtic Body, it will rebound from it again with the ſame Velocity with which it ſtruck the Plane.*

For, ſince the parts which are ſtruck, of the elaſtic body, ſuddenly yield and give way by the force of the blow, and as ſuddenly reſtore themſelves again with a force equal to the force which impreſſed them, by the definition of elaſtic bodies; the intenſity of the action of that reſtoring force on the plane, will be equal to the force or momentum with which the body ſtruck the plane. And, as action and reaction are equal and contrary, the plane will act with the ſame force on the body, and ſo cauſe it to rebound or move back again with the ſame velocity as it had before the ſtroke.

But hard or ſoft bodies, being devoid of elaſticity, by the definition, having no reſtoring force to throw them off again, they muſt neceſſarily adhere to the plane ſtruck.

60. *Corol.* 1. The effect of the blow of the elaſtic body, on the plane, is double to that of the unelaſtic one, the velocity and maſs being equal in each.

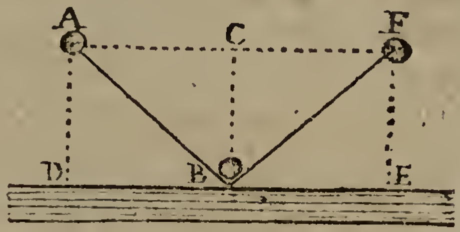
For the force of the blow from the unelaſtic body, is as its maſs and velocity, which is only deſtroyed by the reſiſtance of the plane. But in the elaſtic body, that force is not only deſtroyed and ſuſtained by the plane; but another alſo equal to it is ſuſtained by the plane, in conſequence of the reſtoring force, and by virtue of which the body is thrown back again with an equal velocity. And therefore the intenſity of the blow is doubled.

61. *Corol.* 2. Hence unelaſtic bodies loſe, by their colliſion, only half the motion loſt by elaſtic bodies; their maſs and velocities being equal.—For the latter communicate double the motion of the former.

PRO-

PROPOSITION XIII.

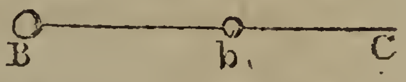
62. *If an Elastic Body* A *impinge on a Firm Plane* DE *at the Point* B, *it will rebound from it in an Angle equal to that in which it struck it; or the Angle of Incidence will be Equal to the Angle of Reflection; namely, the Angle* ABD *equal to the Angle* FBE.



LET AB express the force of the body A in the direction AB; which let be resolved into the two AC, CB, parallel and perpendicular to the plane.—Take BE and CF equal to AC, and draw BF. Now, action and re-action being equal, the plane will resist the direct force CB by another BC equal to it, and in a contrary direction; whereas the other AC, being parallel to the plane, is not acted on nor diminished by it, but still continues as before. The body is therefore reflected from the plane by two forces BC, BE, perpendicular and parallel to the plane, and therefore moves in the diagonal BF by composition. But, because AC is equal to BE or CF, and that BC is common, the two triangles BCA, BCF are mutually similar and equal; and consequently the angles at A and F are equal, as also their equal alternate angles ABD, FBE, which are the angles of incidence and reflection.

PROPOSITION XIV.

63. *To determine the Motion of Non-elastic Bodies, when they strike each other Directly, or in the Same Line of Direction.*



LET the non-elastic body B, moving with the velocity V in the direction B*b*, and the body *b* with the velocity *v*, strike each other. Then, because the momentum of any moving body is as the mass into the velocity, $BV = M$ is the momentum of the body B, and $bv = m$ the momentum of the body *b*, which let be the less powerful of the two motions. Then, by prop. 10, the bodies will both move together as one mass in the direction BC after the stroke, whether before the stroke the body *b* moved towards C or towards B. Now, according as that motion of *b* was from or towards B, that is, whether the motions were in the same or contrary ways, the momentum after the stroke, in direction BC, will be

be the ſum or difference of the momentums before the ſtroke; namely, the momentum in direction BC will be

$BV + bv$, if the bodies moved the ſame way, or
$BV - bv$, if they moved contrary ways, and
BV only, if the body b were at reſt.

Then divide each momentum by the common maſs of matter $B + b$, and the quotient will be the common velocity after the ſtroke in the direction BC; namely, the common velocity will be,

$\frac{BV + bv}{B + b}$ in the firſt caſe, $\frac{BV - bv}{B + b}$ in the 2nd, and $\frac{BV}{B + b}$ in the third.

64. For example, if the bodies, or weights, B and b be as 5 to 3, and their velocities V and v, as 6 to 4, or as 3 to 2, before the ſtroke; then 15 and 6 will be as their momentums, and 8 the ſum of their weights; conſequently after the ſtroke the common velocity will be as

$\frac{15 + 6}{8} = \frac{21}{8}$ or $2\frac{5}{8}$ in the firſt caſe,

$\frac{15 - 6}{8} = \frac{9}{8}$ or $1\frac{1}{8}$ in the ſecond, and

$\frac{15}{8}$ - - - or $1\frac{7}{8}$ in the third.

PROPOSITION XV.

65. *If two Perfectly Elaſtic Bodies impinge on one another; their Relative Velocity will be the ſame both Before and After the Impulſe; that is, they will recede from each other with the Same Velocity with which they approached and met.*

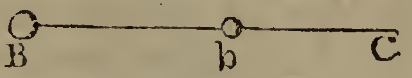
For the compreſſing force is as the intenſity of the ſtroke; which, in given bodies, is as the relative velocity with which they meet or ſtrike. But perfectly elaſtic bodies reſtore themſelves to their former figure by the ſame force by which they were compreſſed; that is, the reſtoring force is equal to the compreſſing force, or to the force with which the bodies approach each other before the impulſe. But the bodies are impelled from each other by this reſtoring force; and therefore this force, acting on the ſame bodies, will produce a relative velocity equal to that which they had before; or it will make the bodies recede from each other with the ſame velocity

velocity with which they before approached, or ſo as to be equally diſtant from one another at equal times before and after the impact.

66. *Remark.* It is not meant by this propoſition, that each body will have the ſame velocity after the impulſe as it had before; for that will be varied according to the relation of the maſſes of the two bodies; but that the velocity of the one will be, after the ſtroke, as much increaſed as that of the other is decreaſed, in one and the ſame direction. So if the elaſtic body B move with a velocity V, and overtake the elaſtic body *b* moving the ſame way with the velocity v; then their relative velocity, or that with which they ſtrike, is $V - v$, and it is with this ſame velocity that they ſeparate from each other after the ſtroke. But if they meet each other, or the body *b* move contrary to the body B; then they meet and ſtrike with the velocity $V + v$, and it is with the ſame velocity that they ſeparate and recede from each other after the ſtroke. But whether they move forward or backward after the impulſe, and with what particular velocities, are circumſtances that depend on the various maſſes and velocities of the bodies before the ſtroke, and which make the ſubject of the next propoſition.

PROPOSITION XVI.

67. *To determine the Motions of Elaſtic Bodies after Striking each other directly.*



LET the elaſtic body B move in the direction BC, with the velocity V; and let the velocity of the other body *b* be v in the ſame direction; which latter velocity v will be poſitive if *b* move the ſame way as B, but negative if *b* move in the oppoſite direction to B. Then their relative velocity in the direction BC is $V - v$; alſo the momenta before the ſtroke are BV and bv, the ſum of which is $BV + bv$ in the direction BC.

Again, put x for the velocity of B, and y for that of *b*, in the ſame direction BC, after the ſtroke; then their relative velocity is $y - x$, and the ſum of their momenta $Bx + by$ in the ſame direction.

But the momenta before and after the colliſion, eſtimated in the ſame direction, are equal, by prop. 10, as alſo the relative velocities, by the laſt prop. Whence ariſe theſe two equations,

$$BV + bv = Bx + by,$$
$$V - v = y - x;$$

the refolution of which equations gives

$$x = \frac{(B - b)V + 2bv}{B + b}, \text{ the velocity of B,}$$
$$y = \frac{-(B - b)v + 2BV}{B + b}, \text{ the velocity of } b.$$

both in the direction BC, when V and v are both pofitive, or the bodies both moved towards C before the collifion. But if v be negative, or the body b moved in the contrary direction before collifion, or towards B; then, changing the fign of v, the fame theorems become

$$x = \frac{(B - b)V - 2bv}{B + b}, \text{ the velocity of B,}$$
$$y = \frac{(B - b)v + 2BV}{B + b}, \text{the veloc. of } b, \text{ in the direction BC.}$$

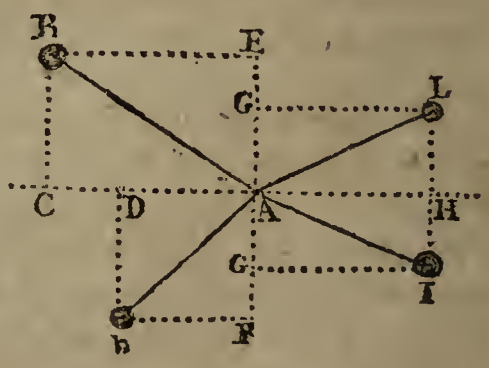
And if b were at reft before the impact, making its velocity $v = 0$, the fame theorems give

$$x = \frac{B - b}{B + b}V, \text{ and } y = \frac{2B}{B + b}V,$$

for the velocities in this cafe.

PROPOSITION XVII.

68. *If Bodies ftrike one another Obliquely, it is propofed to determine their Motions after the Stroke.*



Let the two bodies B, b, move in the oblique directions BA, bA, and ftrike each other at A, with velocities which are in proportion to the lines BA, bA; to find their motions after the impact. Let CAH reprefent the plane in which the bodies touch in the point of concourfe; to which draw the perpendiculars BC, bD, and complete the rectangles CE, DF. Then the motion in BA is refolved into the two BC, CA; and the motion in bA is refolved into the two bD, DA; of which the antecedents BC, bD are the velocities with which they directly meet, and the confequents CA, DA are parallel; therefore, by thefe the bodies do not impinge on each other, and confequently the motions, according to thefe directions,

rections, will not be changed by the impulse; so that the velocities with which the bodies meet, are as BC and *b*D, or their equals EA and FA. The motions, therefore, of the bodies B, *b*, directly striking each other with the velocities EA, FA, will be determined by prop. 16 or 14, according as the bodies are elastic or non-elastic; which being done, let AG be the velocity, so determined, of one of them, as A; and since there remains also in the body a force of moving in the direction parallel to BE, with a velocity as BE, make AH equal to BE, and complete the rectangle GH: then the two motions in AH and AG, or HI, are compounded into the diagonal AI, which therefore will be the path and velocity of the body B after the stroke. And after the same manner is the motion of the other body *b* determined after the impact.

THE LAWS OF GRAVITY; THE DESCENT OF HEAVY BODIES; AND THE MOTION OF PROJECTILES IN FREE SPACE.

PROPOSITION XVIII.

69. *All the Properties of Motion delivered in Proposition VI, its Corollaries and Scholium, for Constant Forces, are true in the Motions of Bodies freely descending by their own Gravity; namely, that the Velocities are as the Times, and the Spaces as the Squares of the Times, or as the Squares of the Velocities.*

FOR, since the force of gravity is uniform, and constantly the same, at all places near the earth's surface, or at nearly the same distance from the centre of the earth; and that this is the force by which bodies descend to the surface; they therefore descend by a force which acts constantly and equally; consequently all the motions freely produced by gravity, are as above specified, by that proposition, &c.

SCHOLIUM.

70. Now it has been found, by numberless experiments, that gravity is a force of such a nature, that all bodies, whether light or heavy, fall perpendicularly through equal spaces in the same time, abstracting from the resistance of the air; as lead or gold and a feather, which in an exhausted receiver fall from the top to the bottom in the same time. It is also

 found,

found, that the velocities acquired by deſcending, are in the exact proportion of the times of deſcent: and farther, that the ſpaces deſcended are proportional to the ſquares of the times, and therefore to the ſquares of the velocities. And hence it follows, that the weights, or gravities, of bodies near the ſurface of the earth, are proportional to the quantities of matter contained in them; and that the ſpaces, times, and velocities, generated by gravity, have the relations contained in the three general proportions before laid down. Moreover, as it is found, by accurate experiments, that a body in the latitude of London, falls nearly $16\frac{1}{12}$ feet in the firſt ſecond of time, and conſequently that at the end of that time it has acquired a velocity double, or of $32\frac{1}{6}$ feet, by corol. 1, prop. 6; therefore, if g denote $16\frac{1}{12}$ feet, the ſpace fallen through in one ſecond of time, or $2g$ the velocity generated in that time; then, becauſe the velocities are directly proportional to the times, and the ſpaces to the ſquares of the times; therefore it will be,

as $1'' : t'' :: 2g : 2gt = v$ the velocity,
and $1^2 : t^2 :: g : gt^2 = s$ the ſpace.

So that for the deſcents of gravity, we have theſe general equations, namely,

$$s = gt^2 = \frac{v^2}{4g} = \tfrac{1}{2}tv.$$

$$v = 2gt = \frac{2s}{t} = 2\sqrt{gs}.$$

$$t = \frac{v}{2g} = \frac{2s}{v} = \sqrt{\frac{s}{g}}.$$

$$g = \frac{v}{2t} = \frac{s}{t^2} = \frac{v^2}{4s}.$$

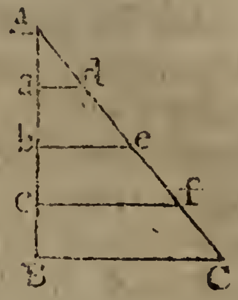
Hence, becauſe the times are as the velocities, and the ſpaces as the ſquares of either, therefore,

if the times be as the numbs.	1,	2,	3,	4,	5,	&c,
the velocities will alſo be as	1,	2,	3,	4,	5,	&c,
and the ſpaces as their ſquares	1,	4,	9,	16,	25,	&c,
and the ſpace for each time as	1,	3,	5,	7,	9,	&c,

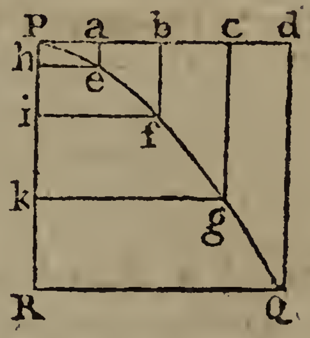
namely, as the ſeries of the odd numbers, which are the differences of the ſquares denoting the whole ſpaces. So that, if the firſt ſeries of natural numbers be ſeconds of time,

namely, the times in ſeconds	$1''$,	$2''$,	$3''$,	$4''$,	&c,
the velocity in feet will be	$32\frac{1}{6}$,	$64\frac{1}{3}$,	$96\frac{1}{2}$,	$128\frac{2}{3}$,	&c,
the ſpaces in the whole times	$16\frac{1}{12}$,	$64\frac{1}{3}$,	$144\frac{3}{4}$,	$257\frac{1}{3}$,	&c,
and the ſpace for each ſecond	$16\frac{1}{12}$,	$48\frac{1}{4}$,	$80\frac{5}{12}$,	$112\frac{7}{12}$,	&c.

71. Theſe



71. These relations of the times, velocities, and spaces, may be aptly represented by certain lines and geometrical figures. Thus, if the line AB denote the time of any body's descent, and BC at right angles to it, the velocity gained at the end of that time, by joining AC, and dividing the time AB into any number of parts at the points a, b, c; then shall ad, be, cf, parallel to BC, be the velocities at the points of time a, b, c, or at the ends of the times Aa, Ab, Ac; because these latter lines, by similar triangles, are proportional to the former ad, be, cf, and the times are proportional to the velocities. Also, the area of the triangle ABC will represent the space descended by the force of gravity in the time AB, in which it generates the velocity BC; because that area is equal to $\frac{1}{2}AB \times BC$, and the space descended is $s = \frac{1}{2}tv$, or half the product of the time and the last velocity. And, for the same reason, the less triangles Aad, Abe, Acf, will represent the several spaces described in the corresponding times Aa, Ab, Ac, and velocities ad, be, cf; those triangles or spaces being also as the squares of their like sides Aa, Ab, Ac, which represent the times, or of ad, be, cf, which represent the velocities.



72. But as areas are rather unnatural representations of the spaces passed over by a body in motion, which are lines, the relations may better be represented by the abscisses and ordinates of a parabola. Thus, if PQ be a parabola, PR its axis, and RQ its ordinate; and Pa, Pb, Pc, &c, parallel to RQ, represent the times from the beginning, or the velocities, then ae, bf, cg, &c, parallel to the axis PR, will represent the spaces described by a falling body in those times; for, in a parabola, the abscisses Ph, Pi, Pk, &c, or ae, bf, cg, &c, which are the spaces described, are as the squares of the ordinates he, if, kg, &c, or Pa, Pb, Pc, &c, which represent the times or velocities.

73. And because the laws for the destruction of motion, are the same as those for the generation of it, by equal forces, but acting in a contrary direction; therefore,

1st, A body thrown directly upwards, with any velocity, will lose equal velocities in equal times.

2d, If

2*d*, If a body be projected upwards, with the velocity it acquired in any time by descending freely; it will lose all its velocity in an equal time, and will ascend just to the same height from whence it fell, and will describe equal spaces in equal times, in rising and falling, but in an inverse order; and it will have equal velocities at any one and the same point of the line described, both in ascending and descending.

3*d*, If bodies be projected upwards, with any velocities, the height ascended to, will be as the squares of those velocities, or as the squares of the times of ascending, till they lose all their velocities.

74. To illustrate now the rules for the natural descent of bodies by a few examples, let it be required,

1*st*, To find the space descended by a body in 7 seconds of time, and the velocity acquired.

Ans.

2*d*, To find the time of generating a velocity of 100 feet per second, and the whole space descended.

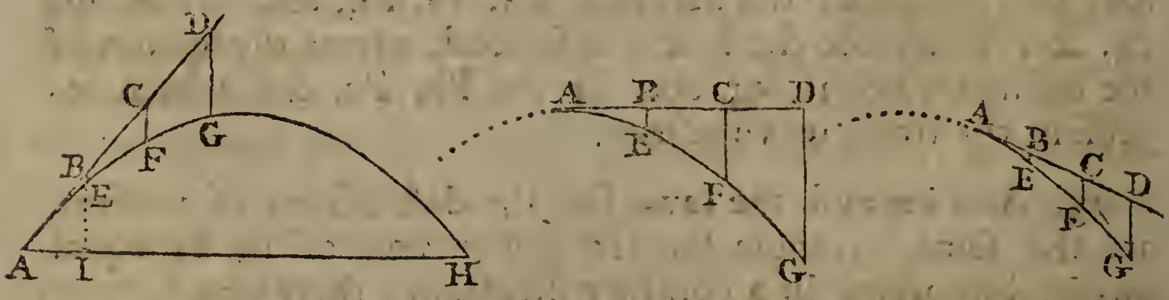
Ans.

3*d*, To find the time of descending 400 feet, and the velocity at the end of that time.

Ans.

PROPOSITION XIX.

75. *If a Body be projected in Free Space, either Parallel to the Horizon, or in an Oblique Direction, by the Force of Gun Powder, or any other Impulse; it will, by this Motion, in Conjunction with the Action of Gravity, describe the Curve Line of a Parabola.*



Let the body be projected from the point A, in the direction AD, with any uniform velocity; then, in any equal portions of time, it would, by prop. 4, describe the equal spaces

ſpaces AB, BC, CD, &c, in the line AD, if it were not drawn continually down below that line by the action of gravity. Draw BE, CF, DG, &c, in the direction of gravity, or perpendicular to the horizon, and equal to the ſpaces through which the body would deſcend by its gravity, in the ſame times in which it would uniformly paſs over the correſponding ſpaces AB, AC, AD, &c, by the projectile motion. Then, ſince by theſe two motions the body is carried over the ſpace AB, in the ſame time as over the ſpace BE, and the ſpace AC in the ſame time as the ſpace CF, and the ſpace AD in the ſame time as the ſpace DG, &c; therefore by the compoſition of motions, at the end of thoſe times, the body will be found reſpectively in the points E, F, G, &c; and conſequently the real path of the projectile will be the curve line AEFG, &c. But the ſpaces AB, AC, AD, &c, deſcribed by uniform motion, are as the times of deſcription; and the ſpaces BE, CF, DG, &c, deſcribed in the ſame times by the accelerating force of gravity, are as the ſquares of the times; conſequently the perpendicular deſcents are as the ſquares of the ſpaces in AD, that is BE, CF, DG, &c, are reſpectively proportional to AB^2, AC^2, AD^2, &c; which is the property of the parabola by theor. 8, Con. Sect. Therefore the path of the projectile is the parabolic line AEFG, &c, to which AD is a tangent at the point A.

76. *Corol.* 1. The horizontal velocity of a projectile, is always the ſame conſtant quantity, in every point of the curve; becauſe the horizontal motion is in a conſtant ratio to the motion in AD, which is the uniform projectile motion. And the conſtant horizontal velocity, is in proportion to the projectile velocity, as radius to the coſine of the angle DAH, or angle of elevation or depreſſion of the piece above or below the horizontal line AH.

77. *Corol.* 2. The velocity of the projectile in the direction of the curve, or of its tangent at any point A, is as the ſecant of its angle BAI of direction above the horizon. For the motion in the horizontal direction AI is conſtant, and AI is to AB, as radius to the ſecant of the angle A; therefore the motion at A, in AB, is every where as the ſecant of the angle A.

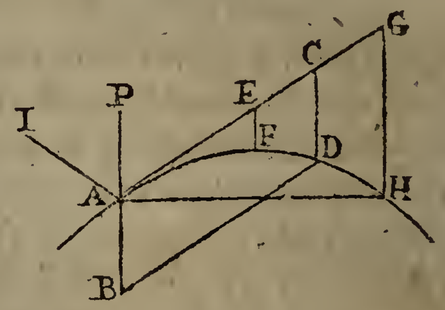
78. *Corol.* 3. The velocity in the direction DG of gravity, or perpendicular to the horizon, at any point G of the curve, is to the firſt uniform projectile velocity at A, or point of contact of a tangent, as 2GD is to AD. For, the times in AD and DG being equal, and the velocity acquired by freely

deſcending

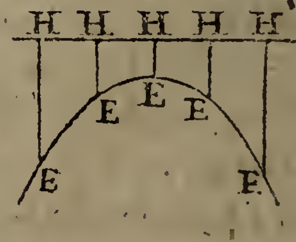
descending through DG being such as would carry the body uniformly over twice DG in an equal time, and the spaces described with uniform motions being as the velocities, therefore the space AD is to the space 2DG, as the projectile velocity at A, to the perpendicular velocity at G.

PROPOSITION XX.

79. *The Velocity in the Direction of the Curve, at any Point of it, as* A, *is equal to that which is generated by Gravity in freely descending through a Space which is equal to One-Fourth of the Parameter of the Diameter of the Parabola at that Point.*



Let PA or AB be the height due to the velocity of the projectile at any point A, in the direction of the curve or tangent AC, or the velocity acquired by falling through that height; and complete the parallelogram ACDB. Then is CD = AB or AP, the height due to the velocity in the curve at A; and CD is also the height due to the perpendicular velocity at D, which must be equal to the former: but, by the last corol. the velocity at A is to the perpendicular velocity at D, as AC to 2CD; and as these velocities are equal, therefore AC or BD is equal to 2CD, or 2AB; and hence AB or AP is equal to $\frac{1}{2}$BD, or $\frac{1}{4}$ of the parameter of the diameter AB, by corol. to theor. 13 of the Parabola.



80. *Corol.* 1. Hence, and from cor. 2, theor. 13 of the Parabola, it appears that, if from the directrix of the parabola which is the path of the projectile, several lines HE be drawn perpendicular to the directrix, or parallel to the axis; then the velocity of the projectile in the direction of the curve, at any point E, is always equal to the velocity acquired by a body falling freely through the perpendicular line HE.

81. *Corol.* 2. If a body, after falling through the height PA (last fig. but one), which is equal to AB, and when it arrives at A, have its course changed, by reflection from an elastic plane AI, or otherwise, into any direction AC, without altering the velocity; and if AC be taken = 2AP or 2AB, and

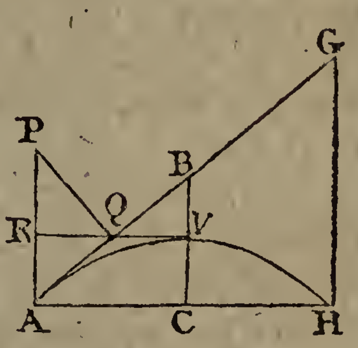
and the parallelogram be completed; then the body will defcribe the parabola paffing through the point D.

82. *Corol.* 3. Becaufe $AC = 2AB$ or $2CD$ or $2AP$, therefore $AC^2 = 2AP \times 2CD$ or $AP . 4CD$; and becaufe all the perpendiculars EF, CD, GH are as AE^2, AC^2, AG^2; therefore alfo $AP . 4EF = AE^2$, and $AP . 4GH = AG^2$, &c; and, becaufe the rectangle of the extremes is equal to the rectangle of the means of four proportionals, therefore always

it is $AP : AE :: AE : 4EF$,
and $AP : AC :: AC : 4CD$,
and $AP : AG :: AG : 4GH$,
and fo on.

PROPOSITION XXI.

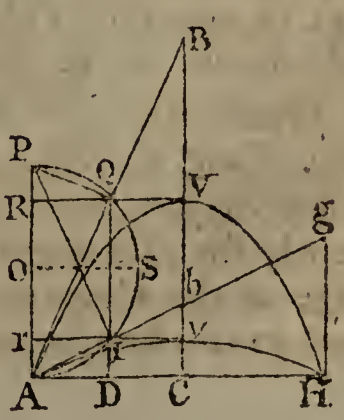
83. *Having given the Direction, and the Impetus, or Altitude due to the Firft Velocity of a Projectile; to determine the Greateft Height to which it will rife, and the Random or Horizontal Range.*



LET AP be the height due to the projectile velocity at A, AG the direction, and AH the horizon. Upon AG let fall the perpendicular PQ, and on AP the perpendicular QR; fo fhall AR be equal to the greateft altitude CV, and 4QR equal to the horizontal range AH. Or, having drawn PQ perp. to AG, take $AG = 4AQ$, and draw GH perp. to AH; then AH is the range.

For, by the laft corollary, $AP : AG :: AG : 4GH$;
and, by fimilar triangles, $AP : AG :: AQ : GH$,
or - $AP : AG :: 4AQ : 4GH$;
therefore $AG = 4AQ$; and, by fimilar triangles, $AH = 4QR$.

Alfo, if V be the vertex of the parabola, then AB or $\frac{1}{2}AG = 2AQ$, or $AQ = QB$; confequently $AR = BV$, which is $= CV$ by the property of the parabola.



84. *Corol.* 1. Becaufe the angle Q is a right angle, which is the angle in a femicircle, therefore if, upon AP, as a diameter, a femicircle be defcribed, it will pafs through the point Q.

85. *Corol.* 2. If the horizontal range and the projectile velocity be

be given, the direction of the piece so as to hit the object H, will be thus easily found: Take AD = $\frac{1}{4}$AH, draw DQ perpendicular to AH, meeting the semicircle, described on the diameter AP, in Q and q; then AQ or Aq will be the direction of the piece. And hence it appears, that there are two directions AB, Ab, which, with the same projectile velocity, give the very same horizontal range AH. And these two directions make equal angles qAD, QAP with AH and AP, because the arc PQ = the arc Aq.

86. *Corol.* 3. Or, if the range AH, and direction AB, be given; to find the altitude and velocity or impetus. Take AD = $\frac{1}{4}$AH, and erect the perpendicular DQ, meeting AB in Q; so shall DQ be equal to the greatest altitude CV. Also, erect AP perpendicular to AH, and QP to AQ; so shall AP be the height due to the velocity.

87. *Corol.* 4. When the body is projected with the same velocity, but in different directions: the horizontal ranges AH will be as the sines of double the angles of elevation.—Or, which is the same, as the rectangle of the sine and cosine of elevation. For AD or RQ, which is $\frac{1}{4}$AH, is the sine of the arc AQ, which measures double the angle QAD of elevation.

And when the direction is the same, but the velocities different; the horizontal ranges are as the square of the velocities, or as the height AP, which is as the square of the velocity; for the sine AD or RQ or $\frac{1}{4}$AH is as the radius, or as the diameter AP.

Therefore, when both are different, the ranges are in the compound ratio of the squares of the velocities, and the sines of double the angles of elevation.

88. *Corol.* 5. The greatest range is when the angle of elevation is 45°, or half a right angle; for the double of 45 is 90, which has the greatest sine. Or the radius OS, which is $\frac{1}{4}$ of the range, is the greatest sine.

And hence the greatest range, or that at an elevation of 45°, is just double the altitude AP which is due to the velocity, or equal to 4VC. And consequently, in that case, C is the focus of the parabola, and AH its parameter. Also, the ranges are equal, at angles equally above and below 45°.

89. *Corol.* 6. When the elevation is 15°, the double of which, or 30°, has its sine equal to half the radius; consequently then its range will be equal to AP, or half the greatest range at the elevation of 45°; that is, the range at 15°, is equal to the impetus or height due to the projectile velocity.

90. *Corol.* 7.

90. *Corol.* 7. The greateſt altitude CV, being equal to AK, is as the verſed ſine of double the angle of elevation, and alſo as AP or the ſquare of the velocity. Or as the ſquare of the ſine of elevation, and the ſquare of the velocity; for the ſquare of the ſine is as the verſed ſine of the double angle.

91. *Corol.* 8. The time of flight of the projectile, which is equal to the time of a body falling freely through GH or 4CV, four times the altitude, is therefore as the ſquare root of the altitude, or as the projectile velocity and ſine of the elevation.

SCHOLIUM.

92. From the laſt propoſition, and its corollaries, may be deduced the following ſet of theorems, for finding all the circumſtances of projectiles on horizontal planes, having any two of them given. Thus, let s, c, t denote the ſine, coſine, and tangent of elevation; $ſ$, v the ſine and verſed ſine of the double elevation; R the horizontal range, T the time of flight, V the projectile velocity, H the greateſt height of the projectile, $g = 16\frac{1}{12}$ feet, and a the impetus, or the altitude due to the velocity V. Then,

$$R = 2aſ = 4asc = \frac{ſV^2}{2g} = \frac{scV^2}{g} = \frac{gcT^2}{s} = \frac{gT^2}{t} = \frac{4H}{t},$$

$$V = \sqrt{4ag} = \sqrt{\frac{2gR}{ſ}} = \sqrt{\frac{gR}{sc}} = \frac{gT}{s} = \frac{2}{s}\sqrt{gH},$$

$$T = \frac{sV}{g} = 2s\sqrt{\frac{a}{g}} = \sqrt{\frac{tR}{g}} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}},$$

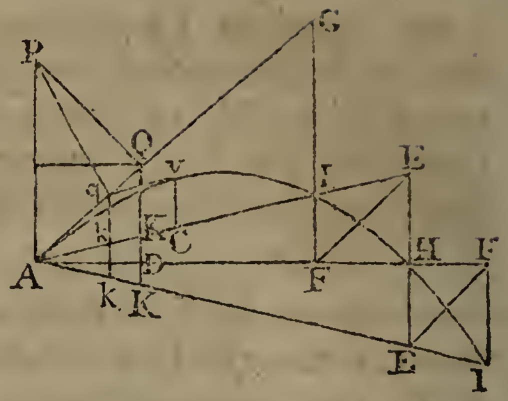
$$H = as^2 = \tfrac{1}{2}av = \tfrac{1}{4}tR = \frac{sR}{4c} = \frac{s^2V^2}{4g} = \frac{vV^2}{8g} = \frac{g}{4}T^2.$$

And from any of theſe, the angle of direction may be found. Alſo, in theſe theorems, g may, in many caſes, be taken $= 16$, without the ſmall fraction $\frac{1}{12}$, which will be near enough for common uſe.

PROPOSITION XXII.

93. *To determine the Range on an Oblique Plane; having given the Impetus or Velocity, and the Angle of Direction.*

Let AE be the oblique plane, at a given angle, either above or below the horizontal plane AH; AG the direction of



of the piece, and AP the altitude due to the projectile velocity at A.

By the last proposition, find the horizontal range AH to the given velocity and direction; draw HE perpendicular to AH, meeting the oblique plane in E; draw EF parallel to AG, and FI parallel to HE; so shall the projectile pass through I, and the range on the oblique plane will be AI. As is evident by theor. 15 of the Parabola, where it is proved, that if AH, AI be any two lines terminated at the curve, and IF, HE parallel to the axis; then is EF parallel to the tangent AG.

94. *Otherwise*, without the Horizontal Range.

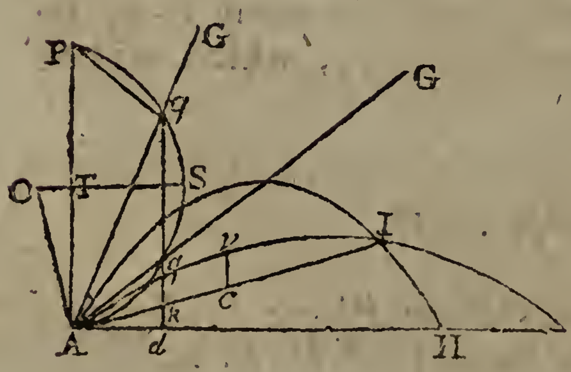
Draw PQ perp. to AG, and QD perp. to the horizontal plane AF, meeting the inclined plane in K; take AE = 4AK, draw EF parallel to AG, and FI parallel to AP or DQ; so shall AI be the range on the oblique plane. For, AH = 4AD, therefore EH is parallel to FI, and so on, as above.

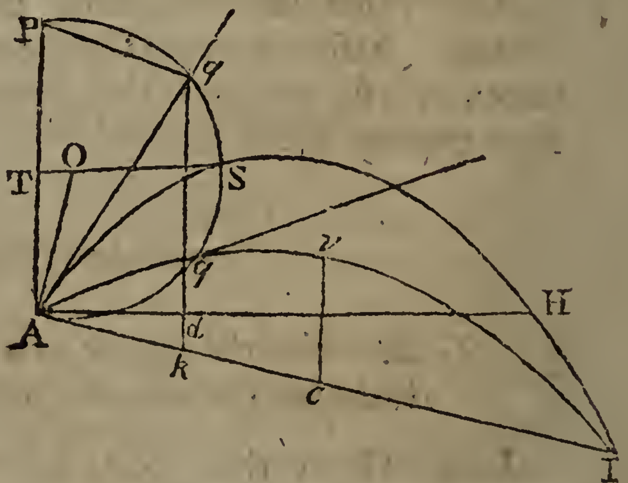
Otherwise,

95. Draw Pq making the angle APq = the angle GAI; then take AG = 4Aq, and draw GI perp. to AH. Or, draw qk perp. to AH, and take AI = 4Ak. Also, kq will be equal to cv the greatest height above the plane.

For, by cor. 2, prop. 20, AP : AG :: AG : 4GI;
and, by sim. triangles, AP : AG :: Aq : GI,
or AP : AG :: 4Aq : 4GI;
therefore AG = 4Aq; and by sim. triangles, AI = 4Ak.

Also, qk, or $\frac{1}{4}$GI, is = to cv by theor. 13 of the Parabola.





96. *Corol.* 1, If AO be drawn perp. to the plane AI, and AP

AP be bisected by the perpendicular STO; then with the centre O describing a circle through A and P, the same will also pass through q, because the angle GAI, formed by the tangent AI and AG, is equal to the angle APq, which will therefore stand on the same arc Aq.

97. *Corol.* 2. If there be given the range and velocity, or the impetus, the direction will hence be easily found thus: Take Ak $= \frac{1}{4}$AI, draw kq perp. to AH, meeting the circle described with the radius AO in two points q and q; then Aq or Aq will be the direction of the piece. And hence it appears, that there are two directions, which, with the same impetus, give the very same range AI. And these two directions make equal angles with AI and AP, because the arc Pq is equal the arc Aq. They also make equal angles with a line drawn from A through S, because the arc Sq is equal the arc Sq.

98. *Corol.* 3. Or, if there be given the range AI, and the direction Aq; to find the velocity or impetus. Take Ak $=$ $\frac{1}{4}$AI, and erect kq perp. to AH, meeting the line of direction in q; then draw qP making the $\angle$ AqP $= \angle$ Akq; so shall AP be the impetus, or the altitude due to the projectile velocity.

99. *Corol.* 4. The range on an oblique plane, with a given elevation, is directly as the rectangle of the cosine of the direction of the piece above the horizon, and the sine of the direction above the oblique plane, and reciprocally as the square of the cosine of the angle of the plane above or below the horizon.

For, put $s =$ sin. $\angle$ qAI or APq,
$c =$ cos. $\angle$ qAH or sin. PAq,
$C =$ cos. $\angle$ IAH or sin. Akd or Akq or AqP.

Then, in the triangle APq,	C : s :: AP : Aq;
and in the triangle Akq,	C : c :: Aq : Ak;
theref. by composition,	C^2 : cs :: AP : Ak $= \frac{1}{4}$AI.

So that the oblique range $AI = \frac{cs}{C^2} \times 4AP$.

100. The range is the greatest when Ak is the greatest; that is, when kq touches the circle in the middle point S; and then the line of direction passes through S, and bisects the angle formed by the oblique plane and the vertex. Also, the ranges are equal at equal angles above and below this direction for the maximum.

101. *Corol.* 5. The greatest height cv or kq of the projectile,

tile, above the plane, is equal to $\frac{s^2}{c^2} \times AP$. And therefore it is as the impetus and square of the sine of direction above the plane directly, and square of the cosine of the plane's inclination reciprocally.

For - C (sin. AqP) : s (sin. APq) :: AP : Aq,
and C (sin. Akq) : s (sin. kAq) :: Aq : kq,
theref. by comp. $C^2 : s^2$ :: AP : kq.

102. *Corol.* 6. The time of flight in the curve AvI is $= \frac{2s}{C}\sqrt{\frac{AP}{g}}$, where $g = 16\frac{1}{12}$ feet. And therefore it is as the velocity and sine of direction above the plane directly, and cosine of the plane's inclination reciprocally. For the time of describing the curve, is equal to the time of falling freely through GI or 4kq or $\frac{4s}{C^2} \times AP$. Therefore, the time being as the square root of the distance,
$\sqrt{g} : \frac{2s}{C}\sqrt{AP} :: 1'' : \frac{2s}{C}\sqrt{\frac{AP}{g}}$, the time of flight.

SCHOLIUM.

103. From the foregoing corollaries may be collected the following set of theorems, relating to projects made on any given inclined planes, either above or below the horizontal plane. In which the letters denote as before, namely,

$c =$ cos. of direction above the horizon,
$C =$ cos. of inclination of the plane,
$s =$ sin. of direction above the plane,
R the range on the oblique plane,
T the time of flight,
V the projectile velocity,
H the greatest height above the plane,
a the impetus, or alt. due to the velocity V,
$g = 16\frac{1}{12}$ feet. Then,

$$R = \frac{cs}{C^2} \times 4a = \frac{cs}{C^2 g} V^2 = \frac{gc}{s} T^2 = \frac{4c}{s} H.$$

$$H = \frac{s^2}{C^2} a = \frac{s^2 V^2}{4gC^2} = \frac{sR}{4c} = \frac{g}{4} T^2.$$

$$V = \sqrt{4ag} = C\sqrt{\frac{gR}{cs}} = \frac{gC}{s} T = \frac{2C}{s}\sqrt{gH}.$$

$$T = \frac{2s}{C}\sqrt{\frac{a}{g}} = \frac{sV}{gC} = \sqrt{\frac{sR}{gc}} = 2\sqrt{\frac{H}{g}}.$$

And from any of these, the angle of direction may be found.

PRAC-

PRACTICAL GUNNERY.

104. THE two foregoing propofitions contain the whole theory of projectiles, with theorems for all the cafes, regularly arranged for ufe, both for oblique and horizontal planes. But, before they can be applied to ufe in refolving the feveral cafes in the practice of gunnery, it is neceffary that fome more data be laid down, as derived from good experiments made with balls or fhells difcharged from cannon or mortars, by gunpowder, under different circumftances. For, without fuch experiments and data, thofe theorems can be of very little ufe in real practice, on account of the imperfections and irregularities in the firing of gunpowder, and the expulfion of balls from guns, but more efpecially on account of the enormous refiftance of the air to all projectiles that are made with any velocities that are confiderable. As to the cafes in which projectiles are made with fmall velocities, or fuch as do not exceed 200 or 300, or 400 feet, per fecond of time, they may be refolved tolerably near the truth, efpecially for the larger fhells, by the parabolic theory laid down above. But, in cafes of great projectile velocities, that theory is quite inadequate, without the aid of feveral data drawn from many and good experiments. For fo great is the effect of the refiftance of the air to projectiles of confiderable velocity, that fome of thofe which in the air range only between 2 and 3 miles at the moft, would in vacuo range about ten times as far, or between 20 and 30 miles.

The effects of this refiftance are alfo various, according to the velocity, the diameter, and the weight of the projectile. So that the experiments made with one fize of ball or fhell, will not ferve for another fize, though the velocity fhould be the fame; neither will the experiments made with one velocity ferve for other velocities, though the ball be the fame. And therefore it is plain that, to form proper rules for practical gunnery, we ought to have good experiments made with each fize of mortar, and with every variety of charge, from the leaft to the greateft. And not only fo, but thefe ought alfo to be repeated at many different angles of elevation, namely, for every fingle degree between 30° and 60° elevation, and at intervals of 5° above 60° and below 30°, from the vertical direction to point blank. By fuch a courfe of experiments it will be found, that the greateft range, inftead of being conftantly that for an elevation of 45°, as in the parabolic theory, will be at all intermediate degrees between 30 and 45, being

being more or leſs, both according to the velocity and the weight of the projectile; the ſmaller velocities and larger ſhells ranging fartheſt when projected almoſt at an elevation of 45°; while the greateſt velocities, eſpecially with the ſmaller ſhells, range fartheſt with an elevation of about 30°.

105. There have, at different times, been made certain ſmall parts of ſuch a courſe of experiments as is hinted at above. Such as the experiments or practice carried on in the year 1773, on Woolwich Common; in which all the ſizes of mortars were uſed, and a variety of ſmall charges of powder. But they were all at the elevation of 45°; and conſequently theſe are defective in the higher charges, and in all the other angles of elevation.

Other experiments were alſo carried on in the ſame place in the year 1784 and 1786, with various angles of elevation indeed, but with only one ſize of mortar, and only one charge of powder, and that but a ſmall one too: ſo that all thoſe nearly agree with the parabolic theory. Other experiments have alſo been carried on with the balliſtic pendulum, at different times; from which have been obtained ſome of the laws for the quantity of powder, the weight and velocity of the ball, the length of the gun, &c. Namely, that the velocity of the ball varies as the ſquare root of the charge directly, and as the ſquare root of the weight of ball reciprocally; and that, ſome rounds being fired with a medium length of one pounder gun, at 15° and 45° elevation, and with 2, 4, 8, and 12 ounces of powder, gave nearly the velocities, ranges, and times of flight, as they are here ſet down in the following Table.

Powder.	Elevation. of gun.	Velocity of ball.	Range.	Time of flight.
oz.		feet.	feet.	
2	15°	860	4100	9″
4	15	1230	5100	12
8	15	1640	6000	$14\frac{1}{2}$
12	15	1680	6700	$15\frac{1}{2}$
2	45	860	5100	21

106. But as we are not yet provided with a ſufficient number and variety of experiments, on which to eſtabliſh true rules for practical gunnery, independent of the parabolic theory, we muſt content ourſelves with the data of ſome one certain

certain experimented range and time of flight, at a given angle of elevation; and then by help of these, and the rules in the parabolic theory, determine the like circumstances for other elevations that are not greatly different from the former, as in the following examples:

107. *Example* 1. If a ball of 1lb. acquire a velocity of 1600 feet per second, when fired with 8 ounces of powder; it is required to find with what velocity each of the several kinds of shells will be discharged by the full charges of powder, viz.

Nature of the shells in inches	13	10	8	$5\frac{1}{2}$	$4\frac{2}{5}$
Their weight in lbs. - -	196	90	48	16	8
Charge of powder in lbs. -	9	4	2	1	$\frac{1}{2}$
Ans. the velocities are -	485	477	462	566	566

108. *Exam.* 2. If a shell be found to range 1000 yards, when discharged at an elevation of 45°; how far will it range when the elevation is 30° 16′, the charge of powder being the same? Ans. 2612 feet or 871 yards.

109. *Exam.* 3. The range of a shell, at 45° elevation, being found to be 3750 feet: at what elevation must the piece be set, to strike an object at the distance of 2810 feet, with the same charge of powder?

Ans. at 24° 16′, or at 65° 44′.

110. *Exam.* 4. With what impetus, velocity, and charge of powder, must a 13 inch shell be fired, at an elevation of 32° 12′, to strike an object at the distance of 3250 feet?

Ans. impetus 1802, veloc. 340, charge 4lb. $7\frac{1}{2}$ oz.

111. *Exam.* 5. A shell being found to range 3500 feet, when discharged at an elevation of 25° 12′; how far then will it range at an elevation of 36° 15′ with the same charge of powder? Ans. 4332 feet.

112. *Exam.* 6. If, with a charge of 9lb. of powder, a shell range 4000 feet; what charge will suffice to throw it 3000 feet, the elevation being 45° in both cases?

Ans. $6\frac{3}{4}$lb. of powder.

113. *Exam.* 7. What will be the time of flight for any given range, at the elevation of 45°?

Ans. the time in secs. is $\frac{1}{4}$ the sq. root of the range in feet.

114. *Exam.* 8. In what time will a shell range 3250 feet, at an elevation of 32°? Ans. $11\frac{1}{4}$ sec. nearly.

115. *Exam.* 9. How far will a ſhot range on a plane which aſcends 8° 15′, and on another which deſcends 8° 15′; the impetus being 3000 feet, and the elevation of the piece 32° 30′? Anſ. 4244 feet on the aſcent, and 6745 feet on the deſcent.

116. *Exam.* 10. How much powder will throw a 13 inch ſhell 4244 feet on an inclined plane, which aſcends 8° 15′, the elevation of the mortar being 32° 30′?

Anſ. 7·3765lb. or 7lb. 6 oz.

117. *Exam.* 11. At what elevation muſt a 13 inch mortar be pointed, to range 6745 feet, on a plane which deſcends 8° 15′; the charge being $7\frac{3}{8}$lb. of powder? Anſ. 32° 30′.

118. *Exam.* 12. In what time will a 13 inch ſhell ſtrike a plane which riſes 8° 30′, when elevated 45°, and diſcharged with an impetus of 2304 feet? Anſ. $14\frac{2}{5}$ ſeconds.

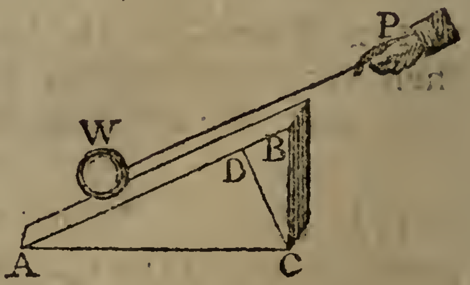
THE DESCENT OF BODIES ON INCLINED PLANES AND CURVE SURFACES.—THE MOTION OF PENDULUMS.

PROPOSITION XXIII.

119. *If a Weight* W *be ſuſtained on an Inclined Plane* AB, *by a Power* P, *acting in a Direction* WP, *parallel to the Plane. Then*

The Weight of the Body, W,	*The Length* AB,
The Suſtaining Power, P, *and*	*The Height* BC, *and*
The Preſſure on the Plane, p,	*The Baſe* AC,
are reſpectively as	*of the Plane.*

FOR, draw CD perpendicular to the plane. Now here are three forces, keeping one another in equilibrio; namely, the weight, or force of gravity, acting perpendicular to AC, or parallel to BC;



the power acting parallel to DB; and the preſſure perpendicular to AB, or parallel to DC: but when three forces keep one another in equilibrio, they are proportional to the ſides of the triangle CBD, made by lines

lines in the direction of those forces, by prop. 8; therefore those forces are to one another as CB, BD, DC. But the two triangles ABC, CBD are equiangular, and have their like sides proportional; therefore the three CB, BD, DC, are to one another respectively as the three AB, BC, AC; which therefore are as the three forces W, P, p.

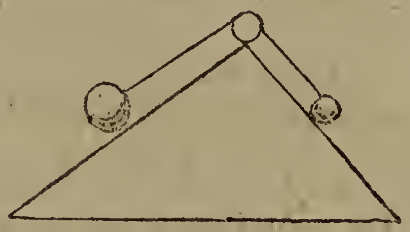
120. *Corol.* 1. Hence the weight W, power P, and pressure p, are respectively as radius, sine, and cosine, of the plane's elevation BAC above the horizon.

For, since the sides of triangles are as the sines of their opposite angles, therefore the three AB, BC, AC, are respectively as - - sin. C, sin. A, sin. B, or as - - - - radius, sine, cosine, of the angle A of elevation.

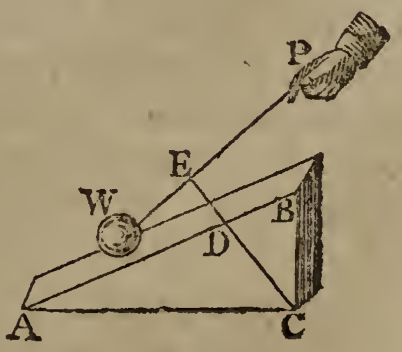
Or, the three forces are as AC, CD, AD; perpendicular to their directions.

121. *Corol.* 2. The power or relative weight that urges a body W down the inclined plane, is $= \frac{BC}{AB} \times W$; or the force with which it descends, or endeavours to descend, is as the sine of the angle A of inclination.

122. *Corol.* 3. Hence, if there be two planes of the same height, and two bodies be laid upon them which are proportional to the lengths of the planes; they will have an equal tendency to descend down the planes. And consequently they will mutually sustain each other if they be connected by a string acting parallel to the planes.

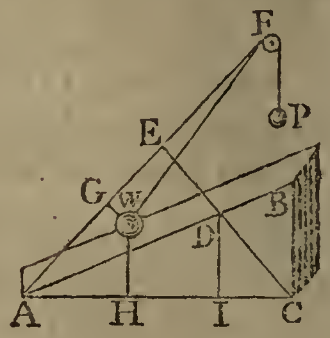


123. *Corol.* 4. In the same manner, when the power P acts in any other direction whatever, WP; by drawing CDE perpendicular to the direction WP, the three forces in equilibrio, namely, the weight W, the power P, and the pressure on the plane, will still be respectively as AC, CD, AD, drawn perpendicular to the direction of those forces.



PROPOSITION XXIV.

124. *If a Weight* W *on an Inclined Plane* AB, *be in Equilibrio with another Weight* P *hanging freely; then if they be set a moving, their Perpendicular Velocities, in that Place, will be Reciprocally as those Weights.*



Let the weight W descend a very small space, from W to A, along the plane, by which the string PFW, will come into the position PFA. Draw WH perpendicular to the horizon AC, and WG perpendicular to AF: then WH will be the space perpendicularly descended by the weight W; and AG, or the difference between FA and FW, will be the space perpendicularly ascended by the weight P; and their perpendicular velocities are as those spaces WH and AG passed over in those directions, in the same time. Draw CDE perpendicular to AF, and DI perpendicular to AC.

Then,

in the sim. figs, AGWH and AEDI,	AG : WH :: AE : DI;
and in the sim. tri. AEC, DIC,	AC : CD :: AE : DI;
but, by cor. 4, prop. 23,	AC : CD :: W : P;
therefore, by equality, -	AG : WH :: W : P.

That is, their perpendicular spaces, or velocities, are reciprocally as their weights or masses.

125. *Corol.* 1. Hence it follows, that if any two bodies be in equilibrio on two inclined planes, and if they be set a moving, their perpendicular velocity will be reciprocally as their weights. Because the perpendicular weight which sustains the one, would also sustain the other.

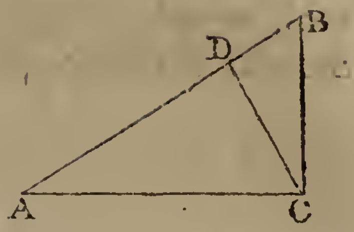
126. *Corol.* 2. And hence also, if two bodies sustain each other in equilibrio, on any planes, and they be put in motion; then each body multiplied by its perpendicular velocity, will give equal products.

PROPOSITION XXV.

127. *The Velocity acquired by a Body descending freely down an Inclined Plane* AB, *is to the Velocity acquired by a Body falling Perpendicularly, in the same Time; as the Height of the Plane* BC, *is to its Length* AB.

For the force of gravity, both perpendicularly and on the plane,

plane, is conſtant; and theſe two, by corol. 2, prop. 23, are to each other as AB to BC. But, by art. 28, the velocities generated by any conſtant forces, in the ſame time, are as thoſe forces. Therefore the velocity down BA is to the velocity down BC, in the ſame time, as the force on BA to the force on BC: that is, as BC to BA.



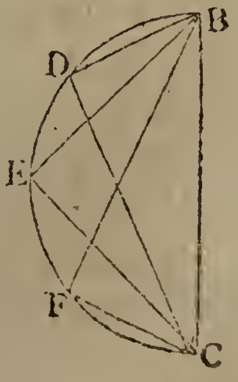
128. *Corol.* 1. Hence, as the motion down an inclined plane is produced by a conſtant force, it will be an uniformly accelerated motion; and therefore the laws before laid down for accelerated motions in general, hold good for motions on inclined planes; ſuch, for inſtance, as the following: That the velocities are as the times of deſcending from reſt; that the ſpaces deſcended are as the ſquares of the velocities, or ſquares of the times; and that, if a body be thrown up an inclined plane, with the velocity it acquired in deſcending, it will loſe all its motion, and aſcend to the ſame height, in the ſame time, and will repaſs any point of the plane with the ſame velocity as it paſſed it in deſcending.

129. *Corol.* 2. Hence alſo, the ſpace deſcended down an inclined plane, is to the ſpace deſcended perpendicularly, in the ſame time, as the height of the plane CB, to its length AB, or as the ſine of inclination to radius. For the ſpaces deſcribed by any forces, in the ſame time, are as the forces, or as the velocities.

130. *Corol.* 3. Conſequently the velocities and ſpaces deſcended, by bodies down different inclined planes, are as the ſines of elevation of the planes.

131. *Corol.* 4. If CD be drawn perpendicular to AB; then, while a body falls freely through the perpendicular ſpace BC, another body will, in the ſame time, deſcend down the part of the plane BD. For, by ſimilar triangles, - - BC : BD :: BA : BC, that is, as the ſpaces deſcended, by corol. 2.

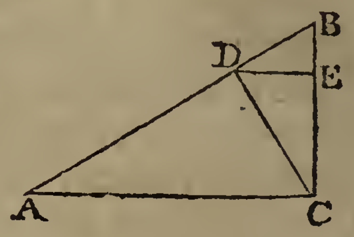
Or, in any right-angled triangle BDC, having its hypothenuſe BC perpendicular to the horizon, a body will deſcend down any of its three ſides BD, BC, DC, in the ſame time. And therefore, if upon the diameter BC a circle be deſcribed, the times of deſcending down any chords BD, BE, BF, DC, EC, FC, &c, will be all equal, and each



each equal to the time of falling freely through the perpendicular diameter BC.

PROPOSITION XXVI.

132. *The Time of deſcending down the Inclined Plane* BA, *is to the Time of falling through the Height of the Plane* BC, *as the Length* BA *to the Height* BC.



Draw CD perpendicular to AB. Then the times of deſcribing BD and BC are equal, by the laſt corol. Call that time t, and the time of deſcribing BA call T.

Now, becauſe the ſpaces deſcribed by conſtant forces, are as the ſquares of the times; therefore $t^2 : T^2 :: BD : BA$.

But the three BD, BC, BA are in continual proportion; therefore $BD : BA :: BC^2 : BA^2$;
hence, by equality, $t^2 : T^2 :: BC^2 : BA^2$,
or $t : T :: BC : BA$.

133. *Corol.* Hence the times of deſcending down different planes, of the ſame height, are to one another as the lengths of the planes.

PROPOSITION XXVII.

134. *A Body acquires the Same Velocity in deſcending down any Inclined Plane* BA, *as by falling perpendicularly through the Height of the Plane* BC.

For, the velocities generated by any conſtant forces, are in the compound ratio of the forces and times of acting. But if we put

F to denote the whole force of gravity in BC,
f the force on the plane AB,
t the time of deſcribing BC, and
T the time of deſcending down AB;

then, by art. 119, $F : f :: BA : BC$;
and by art. 132, $t : T :: BC : BA$;
theref. by comp. $Ft : fT :: 1 : 1$.

That is, the compound ratio of the forces and times, or the ratio of the velocities, is a ratio of equality.

135. *Corol.* 1.

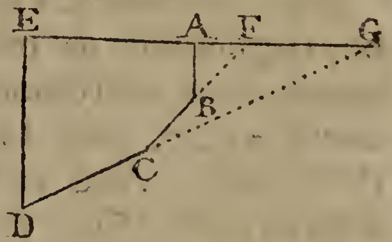
135. *Corol.* 1. Hence the velocities acquired, by bodies descending down any planes, from the same height, to the same horizontal line, are equal.

136. *Corol.* 2. If the velocities be equal, at any two equal altitudes, D, E; they will be equal, at all other equal altitudes A, C.

137. *Corol.* 3. Hence also, the velocities acquired by descending down any planes, are as the square roots of the heights.

PROPOSITION XXVIII.

138. *If a Body descend down any Number of Contiguous Planes* AB, BC, CD; *it will at last acquire the Same Velocity, as a Body falling perpendicularly through the Same Height* ED, *supposing the Velocity not altered by changing from one Plane to another.*



Produce the planes DC, CB to meet the horizontal line EA produced in F and G. Then, by art. 135, the velocity at B is the same, whether the body descend through AB or FB. And therefore the velocity at C will be the same, whether the body descend through ABC or through FC, which is also again, by art. 135, the same as by descending through GC. Consequently it will have the same velocity at D, by descending through the planes AB, CD, as by descending through the plane GD; supposing no obstruction to the motion by the body impinging on the planes at B and C: and this again is the same velocity as by descending through the same perpendicular height ED.

139. *Corol.* 1. If the lines ABCD, &c, be supposed indefinitely small, they will form a curve line, which will be the path of the body; from which it appears, that a body acquires also the same velocity in descending along any curve, as in falling perpendicularly through the same height.

140. *Corol.* 2. Hence also, bodies acquire the same velocity, by descending from the same height, whether they descend perpendicularly, or down any planes, or down any curve or curves. And if their velocities be equal, at any one height, they will be equal at all other equal heights. Therefore the velocity acquired by descending down any lines or curves, are as the square roots of the perpendicular heights.

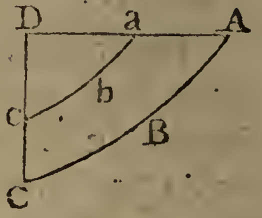
141. *Corol.*

141. *Corol.* 3. And a body, after its defcent through any curve, will acquire a velocity which will carry it to the fame height through an equal curve, or through any other curve, either by running up the fmooth concave fide, or by being retained in the curve by a ftring, and vibrating like a pendulum: Alfo, the velocities will be equal, at all equal altitudes; and the afcent and defcent will be performed in the fame time, if the curves be the fame.

PROPOSITION XXIX.

142. *The Times in which Bodies defcend through Similar Parts of Similar Curves,* ABC, abc, *placed alike, are as the Square Roots of their Lengths.*

That is, the time in AC is to the time in ac, as $\sqrt{AC}$ to $\sqrt{ac}$.



For, as the curves are fimilar, they may be confidered as made up of an equal number of correfponding parts, which are every where, each to each, proportional to the whole. And as they are placed alike, the correfponding fmall fimilar parts will alfo be parallel to each other. But the time of defcribing each of thefe pairs of correfponding parallel parts, by art. 128, are as the fquare roots of their lengths, which, by the fuppofition, are as $\sqrt{AC}$ to $\sqrt{ac}$, the roots of the whole curves. Therefore, the whole times are in the fame ratio of $\sqrt{AC}$ to $\sqrt{ac}$.

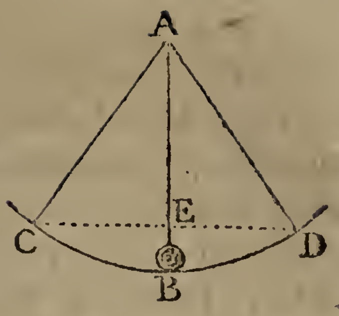
143. *Corol.* 1. Becaufe the axes DC, Dc, of fimilar curves, are as the lengths of the fimilar parts AC, ac; therefore the times of defcent in the curves AC, ac, are as $\sqrt{DC}$ to $\sqrt{Dc}$, or the fquare roots of their axes.

144. *Corol.* 2. As it is the fame thing, whether the bodies run down the fmooth concave fide of the curves, or be made to defcribe thofe curves by vibrating like a pendulum, the lengths being DC, Dc; therefore the times of the vibration of pendulums, in fimilar arcs of any curves, are as the fquare roots of the lengths of the pendulums.

SCHOLIUM.

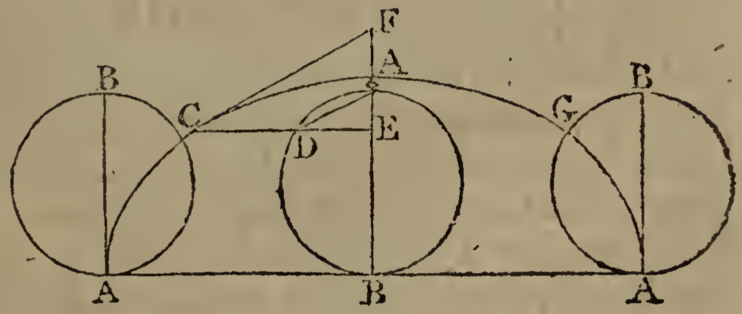
145. Having, in the laft corollary, mentioned the pendulum, it may not be improper here to add fome remarks concerning it.

A pen-



A pendulum confifts of a ball, or any other heavy body B, hung by a fine ftring or thread, moveable about a centre A, and defcribing the arc CBD; by which vibration the fame motion happens to this heavy body, as would happen to any body defcending by its gravity along the fpherical fuperficies CBD, if that fuperficies was perfectly hard and fmooth. If the pendulum be carried to the fituation AC, and then let fall, the ball in defcending will defcribe the arc CB; and in the point B it will have that velocity which is acquired by defcending through CB, or by a body falling freely through EB. This velocity will be fufficient to caufe the ball to afcend through an equal arc BD, to the fame height D from whence it fell at C: having there loft all its motion, it will again begin to defcend by its own gravity; and in the loweft point B it will acquire the fame velocity as before; which will caufe it to reafcend to C: and thus, by afcending and defcending, it will perform continual vibrations in the circumference CBD. And if the motions of pendulums met with no refiftance from the air, and if there were no friction at the centre of motion A, the vibrations of pendulums would never ceafe. But from thofe obftructions, though fmall, it happens, that the velocity of the ball in the point B is a little diminifhed in every vibration; and confequently it does not return precifely to the fame points C or D, but the arcs defcribed continually become fhorter and fhorter, till at length they grow infenfible; unlefs the motion be affifted by a mechanical contrivance, as in clocks, called a maintaining power.

DEFINITION.



146. If the circumference of a circle be rolled on a right line, beginning at any point A, and continued till the fame point A arrive at the line again, making juft one revolution, and thereby meafuring out a ftraight line ABA equal to the circumference of the circle, while the point A in the circumference traces out a curve line ACAGA; then this curve is called a cycloid; and fome of its properties are contained in the following lemma.

LEM-

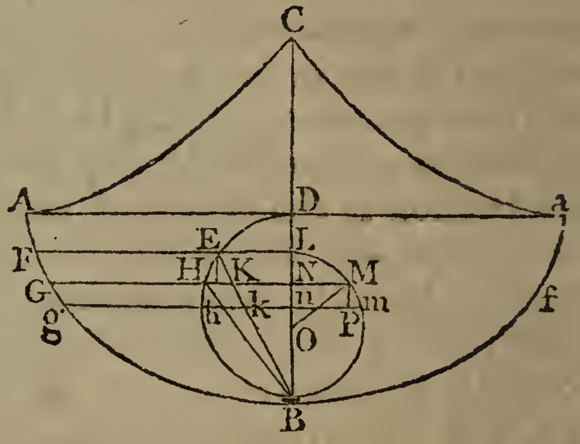
LEMMA.

147. If the generating or revolving circle be placed in the middle of the cycloid, its diameter coinciding with the axis AB, and from any point there be drawn the tangent CF, the ordinate CDE perp. to the axis, and the chord of the circle AD: Then the chief properties are thefe:

The right line CD = the circular arc AD;
The cycloidal arc AC = double the chord AD;
The femi-cycloid ACA = double the diameter AB, and
The tangent CF is parallel to the chord AD.

PROPOSITION XXX.

148. *When a Pendulum vibrates in a Cycloid; the Time of one Vibration, is to the Time in which a Body falls through Half the Length of the Pendulum, as the Circumference of a Circle is to its Diameter.*



Let ABa be the cycloid; DB its axis, or the diameter of the generating femicircle DEB; CB = 2DB the length of the pendulum, or radius of curvature at B. Let the ball defcend from F, and, in vibrating, defcribe the arc FBf. Divide FB into innumerable fmall parts, one of which is Gg; draw FEL, GM, gm perpendicular to DB. On LB defcribe the femi-circle LMB, whofe centre is O; draw Mp parallel to DB; alfo draw the chords BE, BH, EH, and the radius OM.

Now the triangles BEH, BHK are equiangular; therefore BK : BH : : BH : BE, or $BH^2 = BK \cdot BE$, or $BH = \sqrt{BK \cdot BE}$. And the equiangular triangles Mmp, MON, give Mp : Mm : : MN : MO. Alfo, by the nature of the cycloid, Hh is equal and parallel to Gg.

If another body defcend down the chord EB, it will have the fame velocity as the ball in the cycloid has at the fame height. So that Kk and Gg are paffed over with the fame velocity, and confequently the time in paffing them will be as their lengths Gg, Kk, or as Hh to Kk, or BH to BK by fimilar triangles, or $\sqrt{BK \cdot BE}$ to BK, or $\sqrt{BE}$ to $\sqrt{BK}$, or as $\sqrt{BL}$ to $\sqrt{BN}$ by fimilar triangles.

That is, the time in Gg : time in Kk : : $\sqrt{BL}$: $\sqrt{BN}$.

Ag an,

Again, the time of defcribing any fpace with an uniform motion, is directly as the fpace, and reciprocally as the velocity; alfo, the velocity in K or Kk, is to the velocity at B, as $\sqrt{EK}$ to $\sqrt{EB}$, or as $\sqrt{LN}$ to $\sqrt{LB}$; and the uniform velocity for EB is equal to half that at the point B, therefore the time in Kk : time in EB :: $\frac{Kk}{\sqrt{LN}} : \frac{EB}{\frac{1}{2}\sqrt{LB}}$ :: (by fim. tri.)
$\frac{Nn}{\sqrt{LN}} : \frac{LB}{\frac{1}{2}\sqrt{LB}}$ :: Nn or Mp : $2\sqrt{BL \,.\, LN}$.
That is, the time in Kk : time in EB :: Mp : $2\sqrt{BL \,.\, LN}$.
But it was, time in Gg : time in Kk :: $\sqrt{BL} : \sqrt{BN}$; theref. by comp. time in Gg : time in EB :: Mp : $2\sqrt{BN \,.\, NL}$ or 2NM.
But, by fim. tri. Mm : 2OM or BL :: Mp : 2NM.
Theref. time in Gg : time in EB :: Mm : BL.

Confequently the fum of all the times in all the Gg's, is to the time in EB, or the time in DB, which is the fame thing, as the fum of all the Mm's, is to LB;
that is, the time in Fg : time in DB :: Lm : LB,
and the time in FB : time in DB :: LMB : LB,
or the time in FBf : time in DB :: 2LMB : LB.

That is, the time of one whole vibration,
is to the time of falling through half CB,
as the circumference of any circle,
is to its diameter.

149. *Corol.* 1. Hence all the vibrations of a pendulum in a cycloid, whether great or fmall, are performed in the fame time, which time is to the time of falling through the axis, or half the length of the pendulum, as 3·1416 to 1, the ratio of the circumference to its diameter; and hence that time is eafily found thus. Put $p = 3{\cdot}1416$, and l the length of the pendulum, alfo g the fpace fallen by a heavy body in 1″ of time.

then $\sqrt{g} : \sqrt{\frac{1}{2}l} :: 1'' : \sqrt{\frac{l}{2g}}$ the time of falling through $\frac{1}{2}l$,

theref. $1 : p :: \sqrt{\frac{l}{2g}} : p\sqrt{\frac{l}{2g}}$, which therefore is the time of one vibration of the pendulum.

150. And if the pendulum vibrate in a fmall arc of a circle; becaufe that fmall arc nearly coincides with the fmall cycloidal arc at the vertex B; therefore the time of vibration in the fmall arc of a circle, is nearly equal to the time of vibration in the

the cycloidal arc; and confequently the time of vibration in a fmall circular arc, is equal to $p\sqrt{\frac{l}{2g}}$, where l is the radius of the circle.

151. So that, if one of thefe, g or l, be found by experiment, this theorem will give the other. Thus, if g, or the fpace fallen through by a heavy body in 1″ of time, be found, then this theorem will give the length of the feconds pendulum: Or, if the length of the feconds pendulum be obferved by experiment, which is the eafier way; this theorem will give g the defcent of gravity in 1″. Now, in the latitude of London, the length of a pendulum which vibrates feconds, has been found to be $39\frac{1}{8}$ inches; and this being written for l in the theorem, it gives $p\sqrt{\frac{39\frac{1}{8}}{2g}} = 1''$: from hence is found $g = \frac{1}{2}p^2 l = \frac{1}{2}p^2 \times 39\frac{1}{8} = 193{\cdot}07$ inches $= 16\frac{1}{12}$ feet, for the defcent of gravity in 1″; which it has alfo been found to be, very exactly, by many accurate experiments.

SCHOLIUM.

152. Hence is found the length of a pendulum that fhall make any number of vibrations in a given time. Or, the number of vibrations that fhall be made by a pendulum of a given length. Thus, fuppofe it were required to find the length of a half feconds pendulum, or a quarter-feconds pendulum; that is, a pendulum to vibrate twice in a fecond, or 4 times in a fecond. Then, fince the time of vibration is as the fquare root of the length,

therefore $1 : \frac{1}{2} :: \sqrt{39\frac{1}{8}} : \sqrt{l}$,

or $1 : \frac{1}{4} :: 39\frac{1}{8} : \frac{39\frac{1}{8}}{4} = 9\frac{3}{4}$ inches nearly, the length of the half-feconds pendulum.

And $1 : \frac{1}{16} :: 39\frac{1}{8} : 2\frac{4}{9}$ inches, the length of the quarter-feconds pendulum.

Again, if it were required to find how many vibrations a pendulum of 80 inches long will make in a minute. Here $\sqrt{80} : \sqrt{39\frac{1}{8}} :: 60''$ or $1' : 60\sqrt{\frac{39\frac{1}{8}}{80}} = 7\frac{1}{2}\sqrt{31{\cdot}3} =$ - - $41{\cdot}95987$, or almoft 42 vibrations in a minute.

153. In thefe propofitions, the thread is fuppofed to be very fine, or of no fenfible weight, and the ball very fmall, or all the matter united in one point; alfo, the length of the pendulum, is the diftance from the point of fufpenfion, or centre of motion, to this point, or centre of the fmall ball.

ball. But if the ball be large, or the ftring very thick, or the vibrating body be of any other figure; then the length of the pendulum is different, and is meafured, from the centre of motion, not to the centre of magnitude of the body, but to fuch a point, as that if all the matter of the pendulum were collected into it, it would then vibrate in the fame time as the compound pendulum; and this point is called the Centre of Ofcillation; a point which will be treated of in what follows.

THE MECHANICAL POWERS, &c.

154. WEIGHT and Power, when oppofed to each other, fignify the body to be moved, and the body that moves it; or the patient and agent. The power is the agent, which moves, or endeavours to move, the patient or weight.

155. Equilibrium, is an equality of action or force, between two or more powers or weights, acting againft each other, by which they deftroy each other's effects, and remain at reft.

156. Machine, or Engine, is any mechanical inftrument contrived to move bodies. And it is compofed of the mechanical powers.

157. Mechanical Powers, are certain fimple machines, which are commonly employed for raifing greater weights, or overcoming greater refiftances, than could be effected by the natural ftrength without them.—Thefe are ufually accounted fix in number, viz. the Lever and Balance, the Pulley, the Wheel and Axle, the Wedge, the Inclined Plane, and the Screw.

158. Mechanics, is the fcience of forces, and the effects they produce, when applied to machines, in the motion of bodies.

159. Statics, is the fcience of weights, efpecially when confidered in a ftate of equilibrium.

160. Centre of Motion, is the fixed point about which a body moves. And the Axis of Motion, is the fixed line about which it moves.

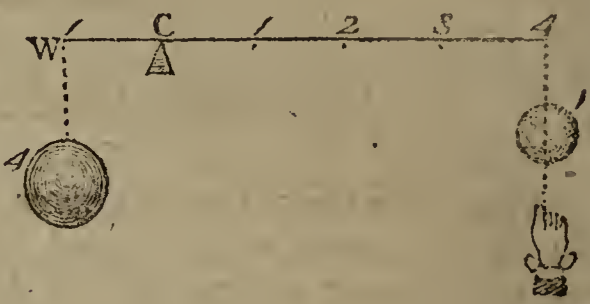
161. Centre of Gravity, is a certain point, upon which a body being freely fufpended, it will reft in any pofition.

OF

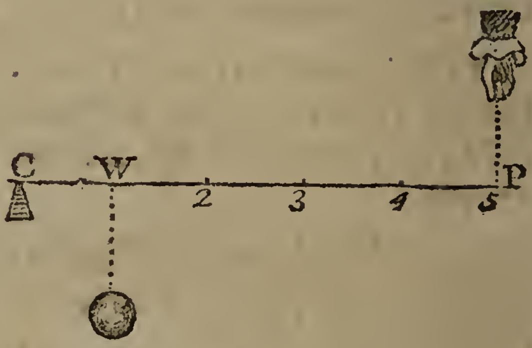
Of the LEVER.

162. A Lever is any inflexible rod, bar, or beam, which ſerves to raiſe weights, while it is ſupported at a point by a fulcrum or prop, which is the centre of motion. The lever is ſuppoſed to be void of gravity or weight, to render the demonſtrations eaſier and ſimpler. There are four kinds of levers.

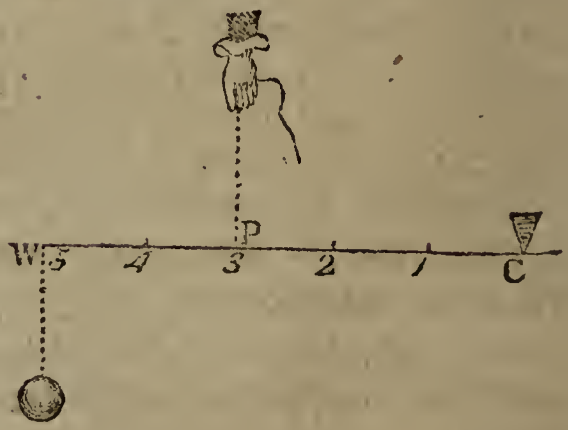
163. A Lever of the Firſt Kind has the prop C between the weight W and the power P. And of this kind are balances, ſcales, crows, hand-ſpikes, ſciſſars, pinchers, &c.



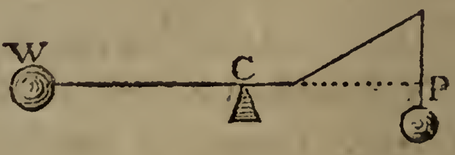
164. A Lever of the Second Kind has the weight between the power and the prop. Such as oars, rudders, cutting knives that are fixed at one end, &c.



165. A Lever of the Third Kind has the power between the weight and the prop. Such as tongs, the bones and muſcles of animals, a man rearing a ladder, &c.



166. A Fourth Kind is ſometimes added, called the Bended Lever. As a hammer drawing a nail.

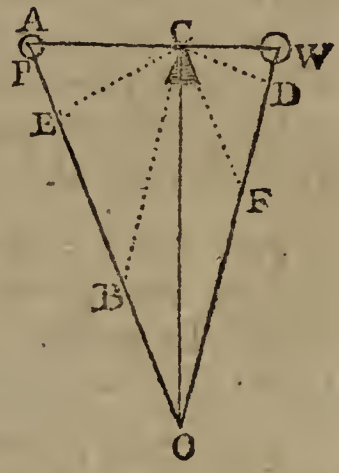


167. In all theſe machines, the power may be repreſented by a weight, which is its moſt natural meaſure, acting downward: but having its direction changed, when neceſſary, by means of a fixed pulley.

PRO-

PROPOSITION XXXI.

168. *When the Weight and Power keep the Lever in Equilibrio, they are to each other Reciprocally as the Distances of their Lines of Direction from the Prop. That is,* P : W :: CD : CE; *where* CD *and* CE *are perpendicular to* WO *and* AO, *which are the Directions of the two Weights, or the Weight and Power* W *and* A.



FOR, draw CF parallel to AO, and CB parallel to WO: Alſo, join CO, which will be the direction of the preſſure on the prop C; for there cannot be an equilibrium unleſs the directions of the three forces all meet in, or tend to, the ſame point, as O. Then, becauſe theſe three forces keep each other in equilibrio, they are proportional to the ſides of the triangle CBO or CFO, which are drawn in the direction of thoſe forces; therefore - - P : W :: CF : FO or CB. But, becauſe of the parallels, the two triangles CDF, CEB are equiangular, therefore - CD : CE :: CF : CB. Hence, by equality, - P : W :: CD : CE.

That is, each force is reciprocally proportional to the diſtance of its direction from the fulcrum.

And it will be found that this demonſtration will ſerve for all the other kinds of levers, by drawing the lines as directed.

169. *Corol.* 1. When the two forces act perpendicularly on the lever, as two weights, &c; then, in caſe of an equilibrium, D coincides with W, and E with P; conſequently then the above proportion becomes P : W :: CW : CP, or the diſtances of the two forces from the fulcrum, taken on the lever, are reciprocally proportional to thoſe forces.

170. *Corol.* 2. If any force P be applied to a lever at A; its effect on the lever, to turn it about the centre of motion C, is as the length of the lever CA, and the ſine of the angle of direction CAE. For the perp. CE is as CA $\times$ s.$\angle$A.

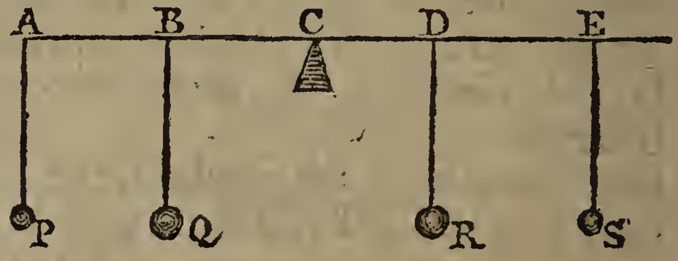
171. *Corol.* 3. Becauſe the product of the extremes is equal to the product of the means, therefore the product of the

the power by the diſtance of its direction, is equal to the product of the weight by the diſtance of its direction.

That is, $P \times CE = W \times CD$.

172. *Corol.* 4. If the lever, with the weight and power fixed to it, be made to move about the centre C; the momentum of the power will be equal to the momentum of the weight; and their velocities will be in reciprocal proportion to each other. For the weight and power will deſcribe circles whoſe radii are the diſtances CD, CE; and ſince the circumferences or ſpaces deſcribed, are as the radii, and alſo as the velocities, therefore the velocities are as the radii CD, CE; and the momenta, which are as the maſſes and velocities, are as the momenta and radii; that is, as $P \times CE$ and $W \times CD$, which are equal by cor. 3.

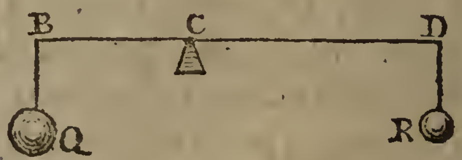
173. *Corol.* 5. In a ſtraight lever, kept in equilibrio by a weight and power acting perpendicularly; then, of theſe three, the power, weight, and preſſure on the prop, any one is as the diſtance of the other two.



174. *Corol.* 6. If ſeveral weights P, Q, R, S, act on a ſtraight lever, and keep it in equilibrio; then the ſum of the products on one ſide of the prop, will be equal to the ſum on the other ſide, made by multiplying each weight by its diſtance; namely,

$P \times AC + Q \times BC = R \times DC + S \times EC$.

For, the effect of each weight to turn the lever, is as the weight multiplied by its diſtance; and in the caſe of an equilibrium, the ſums of the effects, or of the products on both ſides, are equal.



175. *Corol.* 7. Becauſe, when two weights Q and R are in equilibrio, $Q : R :: CD : CB$;

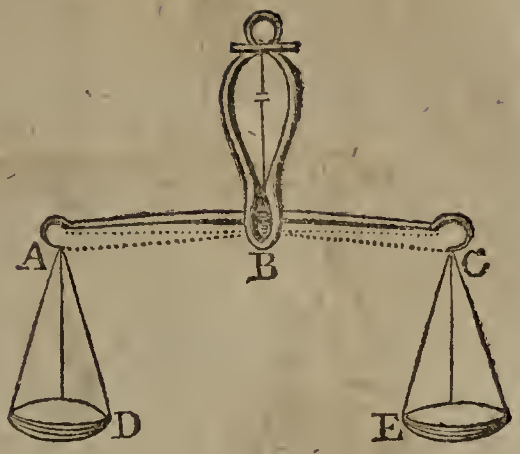
therefore, by compoſition, $Q + R : Q :: BD : CD$,

and $Q + R : R :: BD : CB$.

That is, the ſum of the weights is to either of them, as the ſum of their diſtances is to the diſtance of the other.

SCHO-

SCHOLIUM.



176. On the foregoing principles depends the nature of scales and beams, for weighing all sorts of commodities. For, if the weights be equal, then will the distances be equal also, which gives the construction of the common scales, which ought to have these properties:

1*st*, That the points of suspension of the scales and the centre of motion of the beam, A, B, C, should be in a straight line: 2*d*, That the arms AB, BC be of an equal length: 3*d*, That the centre of gravity be in the centre of motion B, or a little below it: 4*th*, That they be in equilibrio when empty: 5*th*, That there be as little friction as possible at the centre B. A defect in any of these properties, makes the scales either imperfect or false. But it often happens that the one side of the beam is made shorter than the other, and the defect covered by making that scale the heavier, by which means the scales hang in equilibrio when empty; but when they are charged with any weights, so as to be still in equilibrio, those weights are not equal; but the deceit will be detected by changing the weights to the contrary sides, for then the equilibrium will be immediately destroyed.

177. To find the true weight of any body by such a false balance:—First weigh the body in one scale, and afterwards weigh it in the other; then the mean proportional between these two weights, will be the true weight required. For, if any body b weigh W pounds or ounces in the scale D, and only w pounds or ounces in the scale E: then we have these two equations, namely, $AB \cdot b = BC \cdot W$.

and $BC \cdot b = AB \cdot w$;

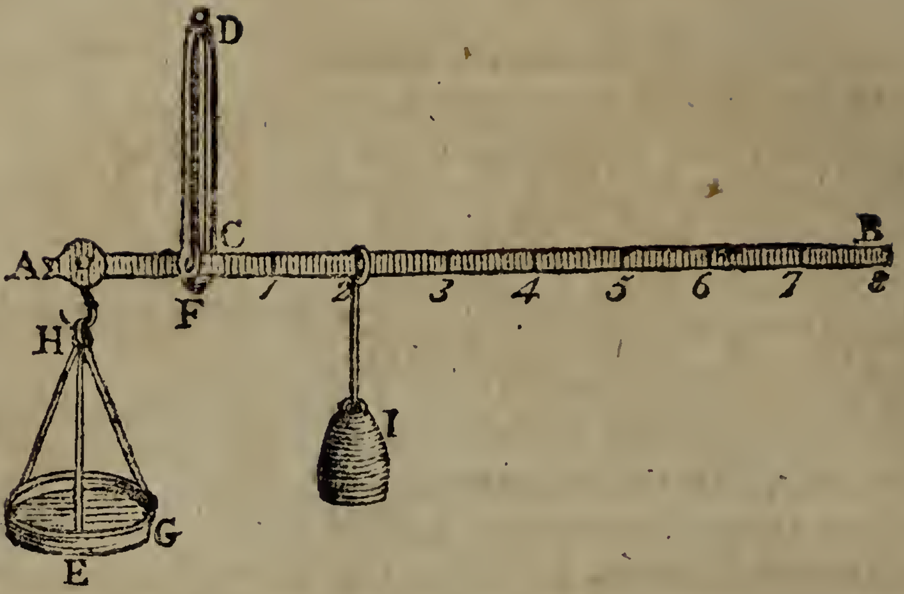
the product of the two is $AB \cdot BC \cdot b^2 = AB \cdot BC \cdot Ww$;

hence then - - - - $b^2 = Ww$,

and - - - $b = \sqrt{Ww}$,

the mean proportional, which is the true weight of the body b.

178. The Roman Statera, or Steelyard, is also a lever, but of unequal brachia or arms, so contrived, that one weight only may serve to weigh a great many, by sliding it backward and forward, to different distances on the longer arm of the lever; and it is thus constructed:



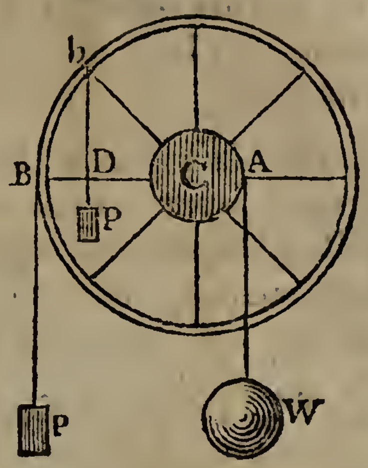
Let AB be the ſteelyard, and C its centre of motion, from whence the diviſions muſt commence if the two arms juſt balance each other: if not, ſlide the conſtant moveable weight I along from B towards C, till it juſt balance the other end without a weight, and there make a notch in the beam, marking it with a cypher o. Then hang on at A a weight W equal to I, and ſlide I back towards B till they balance each other; there notch the beam, and mark it with 1. Then make the weight W double of I, and ſliding I back to balance it, there mark it with 2. Do the ſame at 3, 4, 5, &c, by making W equal to 3, 4, 5, &c, times I; and the beam is finiſhed. Then, to find the weight of any body *b* by the ſteelyard; take off the weight W, and hang on the body *b* at A; then ſlide the weight I backward and forward till it juſt balance the body *b*, which ſuppoſe to be at the number 5; then is *b* equal to 5 times the weight of I. So, if I be one pound, then *b* is 5 pounds; but if I be 2 pounds, then *b* is 10 pounds; and ſo on.

Of the WHEEL and AXLE.

PROPOSITION XXXII.

179. *In the Wheel-and-Axle; the Weight and Power will be in Equilibrio, when the Power* P *is to the Weight* W, *Reciprocally as the Radii of the Circles where they act; that is, as the Radius of the Axle* CA, *where the Weight hangs, to the Radius of the Wheel* CB, *where the Power acts. That is,* P : W :: CA : CB.

HERE the cord, by which the power P acts, goes about the

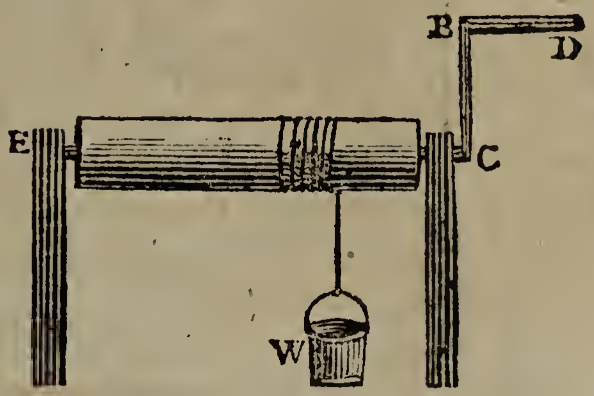


the circumference of the wheel, while that of the weight W goes round its axle, or another ſmaller wheel, attached to the larger, and having the ſame axis or centre C. So that BA is a lever moveable about the point C, the power P acting always at the diſtance BC, and the weight W at the diſtance CA; therefore P : W :: CA : CB.

180. *Corol.* 1. If the wheel be put in motion; then, the ſpaces moved being as the circumferences, or as the radii, the velocity of W will be to the velocity of P, as CA to CB; that is, the weight is moved as much ſlower, as it is heavier than the power; ſo that what is gained in power, is loſt in time. And this is the univerſal property of all machines and engines.

181. *Corol.* 2. If the power do not act at right angles to the radius Cb, but obliquely; draw CD perpendicular to the direction of the power; then, by the nature of the lever, P : W :: CA : CD.

SCHOLIUM.



182. To this power belong all turning or wheel machines, of different radii. Thus, in the roller turning on the axis or ſpindle CE, by the handle CBD; the power applied at B is to the weight W on the roller, as the radius of the roller is to the radius CB of the handle.

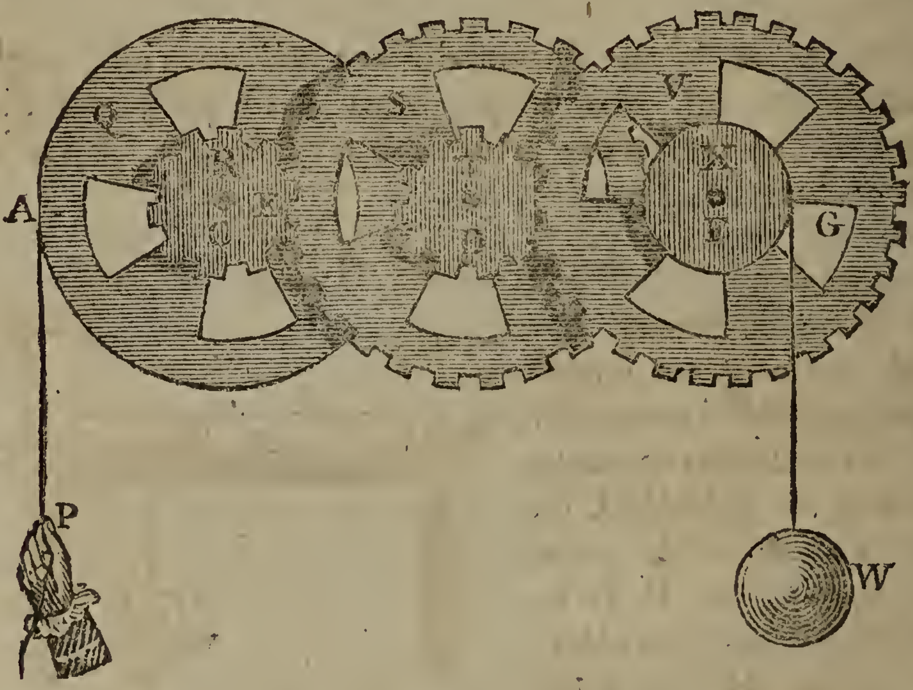
183. And the ſame for all cranes, capſtans, windlaſſes, and ſuch like; the power being to the weight, always as the radius or lever at which the weight acts, to that at which the power acts; ſo that they are always in the reciprocal ratio of their velocities. And to the ſame principle may be referred the gimblet and augur for boring holes.

184. But all this, however, is on ſuppoſition that the ropes or cords, ſuſtaining the weights, are of no ſenſible thickneſs. For, if the thickneſs be conſiderable, or if there be ſeveral folds of them, over one another, on the roller or barrel; then we muſt meaſure to the middle of the outermoſt rope, for the

the radius of the roller; or, to the radius of the roller we muſt add half the thickneſs of the cord, when there is but one fold.

185. The wheel-and-axle has a great advantage over the ſimple lever, in point of convenience. For a weight can be raiſed but a little way by the lever; whereas, by the continual turning of the wheel and roller, the weight may be raiſed to any height, or from any depth.

186. By increaſing the number of wheels too, the power may be multiplied to any extent, making always the leſs wheels to turn greater ones, as far as we pleaſe: and this is commonly called Tooth and Pinion Work, the teeth of one circumference working in the rounds or pinions of another, to turn the wheel. And then, in caſe of an equilibrium, the power is to the weight, as the continual product of the radii of all the axles, to that of all the wheels. So, if the power P



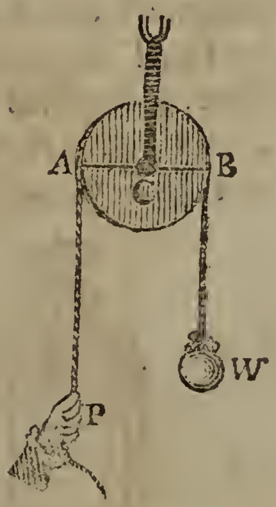
turn the wheel Q, and this turn the ſmall wheel or axle R, and this turn the wheel S, and this turn the axle T, and this turn the wheel V, and this turn the axle X, which raiſes the weight W; then $P : W :: CB . DE . FG : AC . BD . EF$. And in the ſame proportion is the velocity of W ſlower than that of P. Thus, if each wheel be to its axle, as 10 to 1; then $P : W :: 1^3 : 10^3$ or as 1 to 1000. So that a power of one pound will balance a weight of 1000 pounds; but then, when put in motion, the power will move 1000 times faſter than the weight.

OF THE PULLEY.

187. A PULLEY is a ſmall wheel, commonly made of wood or braſs, which turns about an iron axis paſſing through the centre, and fixed in a block, by means of a cord paſſed round its circumference, which ſerves to draw up any weight. The pulley is either ſingle, or combined together, to increaſe the power. It is alſo either fixed or moveable, according as it is fixed to one place, or moves up and down with the weight and power.

PROPOSITION XXXIII.

188. *If a Power ſuſtain a Weight by means of a Fixed Pulley: the Power and Weight are Equal.*



FOR through the centre C of the pulley draw the horizontal diameter AB: then will AB repreſent a lever of the firſt kind, its prop being the fixed centre C; from which the points A and B, where the power and weight act, being equally diſtant, the power P is conſequently equal to the weight W.

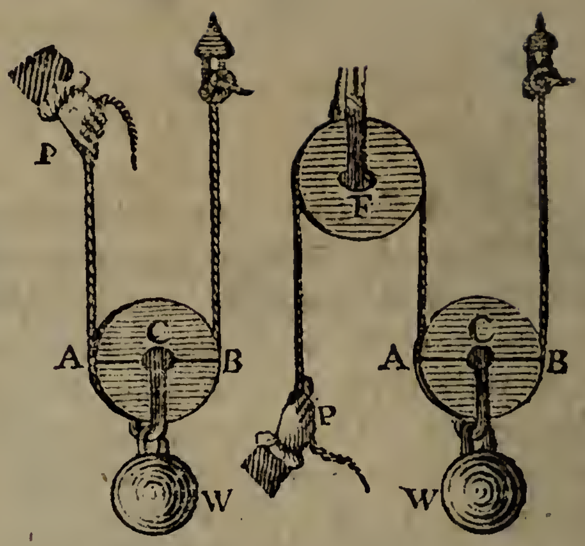
189. *Corol.* Hence, if the pulley be put in motion, the power P will deſcend as faſt as the weight W aſcends. So that the power is not increaſed by the uſe of the fixed pulley, even though the rope go over ſeveral of them. It is, however, of great ſervice in the raiſing of weights, both by changing the direction of the force, for the convenience of acting, and by enabling a perſon to raiſe a weight to any height without moving from his place, and alſo by permitting a great many perſons at once to exert their force on the rope at P, which they could not do to the weight itſelf; as is evident in raiſing the hammer or weight of a pile-driver, as well as on many other occaſions.

PROPOSITION XXXIV.

190. *If a Power ſuſtain a Weight by means of One Moveable Pulley; the Power is but Half the Weight.*

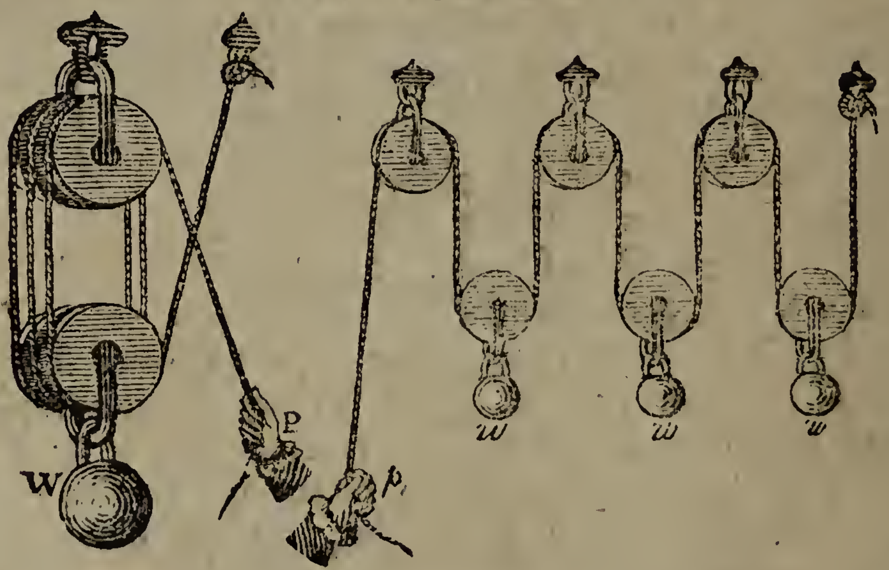
FOR, here AB may be conſidered as a lever of the ſecond kind,

kind, the power acting at A, the weight at C, and the prop or fixed point at B; and because P : W :: CB : AB, and CB = $\frac{1}{2}$AB, therefore P = $\frac{1}{2}$W, or W = 2P.



191. *Corol.* 1. Hence it is evident, that, when the pulley is put in motion, the velocity of the power will be double the velocity of the weight, as the point P descends twice as fast as the point C and weight W rises. It is also evident, that the fixed pulley F makes no difference in the power P, but is only used to change the direction of it, from upwards to downwards.

192. *Corol.* 2. Hence we may estimate the effect of a combination of any number of fixed and moveable pulleys; by which we shall find that every cord going over a moveable pulley, always doubles the power, since each end of the rope bears an equal share of the weight; while each rope that is fixed to a pulley, only increases the power by unity.



Here P = $\frac{1}{6}$W. Here P = $\frac{1}{2}w = \frac{w + w + w}{6}$.

Of the INCLINED PLANE.

193. THE Inclined Plane, is a plane inclined to the horizon, or making an angle with it. It is often reckoned one of the simple mechanic powers; and the double inclined plane makes the wedge. It is employed to advantage in raising heavy bodies in certain situations, diminishing their weights by laying them on the inclined planes.

PRO-

PROPOSITION XXXV.

194. *The Power gained by the Inclined Plane, is in Proportion as the Length of the Plane is to its Height. That is, when a Weight* W *is sustained on an Inclined Plane* BC, *by a Power* P, *acting in the Direction* DW, *parallel to the Plane; then the Weight* W, *is in proportion to the Power* P, *as the Length of the Plane is to its height; that is,* W : P :: BC : AB.

For, draw AE perp. to the plane BC, or to DW. Then we are to consider that the body W is sustained by three forces, viz. 1st, its own weight or the force of gravity, acting perp. to AC, or parallel to BA; 2d, by the power P, acting in the direction WD, parallel to BC, or BE; and 3dly, by the re-action of the planes, perp. to its face, or parallel to the line EA. But when a body is kept in equilibrio by the action of three forces, it has been proved, that the intensities of these forces are proportional to the sides of the triangle, ABE, made by lines drawn in the directions of their actions; therefore those forces are to one another as the three lines - - - AB, BE, AE; that is,
the weight of the body W is as the line AB,
the power P is as the line - BE,
and the pressure on the plane as the line AE.
But the two triangles ABE, ABC are equiangular, and have therefore their like sides proportional; that is,
the three lines - - - - AB, BE, AE,
are to each other respectively as the three BC, AB, AC,
or also as the three - - - BC, AE, CE,
which therefore are as the three forces W, P, p,
where p denotes the pressure on the plane. That is, W : P :: BC : AB, or the weight is to the power, as the length of the plane is to its height.

See more on the Inclined Plane, at pa. 167, &c.

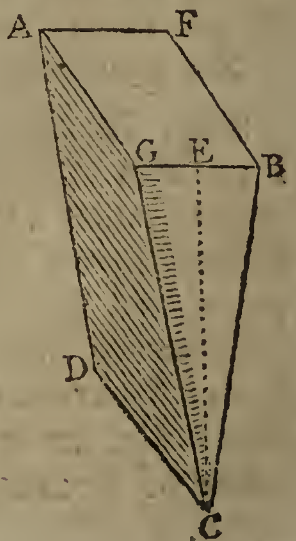
195. *Scholium.* The inclined plane comes into use in some situations in which the other mechanical powers cannot be conveniently applied, or in combination with them. As, in sliding heavy weights either up or down a plank or other plane laid sloping; or letting large casks down into a cellar, or drawing them out of it. Also, in removing earth from a lower situation to a higher, by means of wheel-barrows, or otherwise, as in making fortifications, &c; inclined planes, made of boards, laid aslope, serve for the barrows to run upon.

Of

Of all the various directions of drawing bodies up an inclined plane, or ſuſtaining them on it, the moſt favourable is where it is parallel to the plane BC, and paſſing through the centre of the weight: a direction which is eaſily given to it, by fixing a pulley at D, ſo that a cord paſſing over it, and fixed to the weight, may act or draw parallel to the plane. In every other poſition, it would require a greater power to ſupport the body on the plane, or to draw it up. For if one end of the line be fixed at W, and the other end inclined down towards B, below the direction WD, the body would be drawn down againſt the plane, and the power muſt be increaſed in proportion to the greater difficulty of the traction. And, on the other hand, if the line were carried above the direction of the plane, the power muſt be alſo increaſed; but here only in proportion as it endeavours to lift the body off the plane.

If the length BC of the plane be equal to any number of times its perp. height AB, as ſuppoſe 3 times; then a power P of 1 pound, hanging freely, will balance a weight W of 3 pounds, laid on the plane; and a power P of 2 pounds, will balance a weight W of 6 pounds; and ſo on, always 3 times as much. But then if they be ſet a moving, the perp. deſcent of the power P, will be equal to 3 times as much as the perp. aſcent of the weight W. For, although the weight W aſcends up the direction of the oblique plane BC, juſt as faſt as the power P deſcends perpendicularly, yet the weight riſes only the perp. height AB, while it aſcends up the whole length of the plane BC, which is 3 times as much; that is, for every foot of the perp. riſe of the weight, it aſcends 3 feet up in the direction of the plane, and the power P deſcends juſt as much, or 3 feet.

Of the WEDGE.

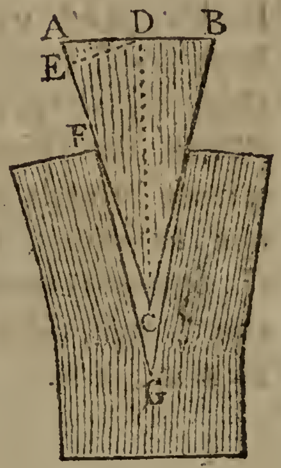


196. THE Wedge is a piece of wood or metal, in form of half a rectangular priſm. AF or BG is the breadth of its back; CE its height; GC, BC its ſides; and its end GBC is compoſed of two equal inclined planes GCE, BCE.

PRO-

PROPOSITION XXXVI.

197. *When a Wedge is in Equilibrio; the Power acting againſt the Back, is to the Force acting Perpendicularly againſt either Side, as the Breadth of the Back* AB, *is to the Length of the Side* AC *or* BC.



FOR, any three forces, which ſuſtain one another in equilibrio, are as the correſponding ſides of a triangle drawn perpendicular to the directions in which they act. But AB is perp. to the force acting on the back, to urge the wedge forward; and the ſides AC, BC are perp. to the forces acting upon them; therefore the three forces are as AB, AC, BC.

198. *Corol.* The force on the back, AB,
Its effect in direct. perp. to AC, AC,
And its effect parallel to AB; DC,
are as the three lines which are perp. to them.

And therefore the thinner a wedge is, the greater is its effect, in ſplitting any body, or in overcoming any reſiſtance againſt the ſides of the wedge.

SCHOLIUM.

199. But it muſt be obſerved, that the reſiſtance, or the forces above mentioned, reſpect one ſide of the wedge only. For if thoſe againſt both ſides be taken in, then, in the foregoing proportions, we muſt take only half the back AD, or elſe we muſt take double the line AC or DC.

In the wedge, the friction againſt the ſides is very great, at leaſt equal to the force to be overcome, becauſe the wedge retains any poſition to which it is driven; and therefore the reſiſtance is doubled by the friction. But then the wedge has a great advantage over all the other powers, ariſing from the force of percuſſion or blow with which the back is ſtruck, which is a force incomparably greater than any dead weight or preſſure, ſuch as is employed in other machines. And accordingly we find it produces effects vaſtly ſuperior to thoſe of any other power; ſuch as the ſplitting and raiſing the largeſt and hardeſt rocks, the raiſing and lifting the largeſt ſhip, by driving a wedge below it, which a man can do by the blow of a mallet: and thus it appears that the ſmall blow of a hammer, on the back of a wedge, is incomparably greater than any mere preſſure, and will overcome it.

OF

Of the SCREW.

200. THE Screw is one of the six mechanical powers, chiefly used in pressing or squeezing bodies close, though sometimes also in raising weights.

The screw is a spiral thread or groove cut round a cylinder, and every where making the same angle with the length of it. So that if the surface of the cylinder, with this spiral thread on it, were unfolded and stretched into a plane, the spiral thread would form a straight inclined plane, whose length would be to its height, as the circumference of the cylinder, is to the distance between two threads of the screw: as is evident by considering that, in making one round, the spiral rises along the cylinder the distance between the two threads.

PROPOSITION XXXVII.

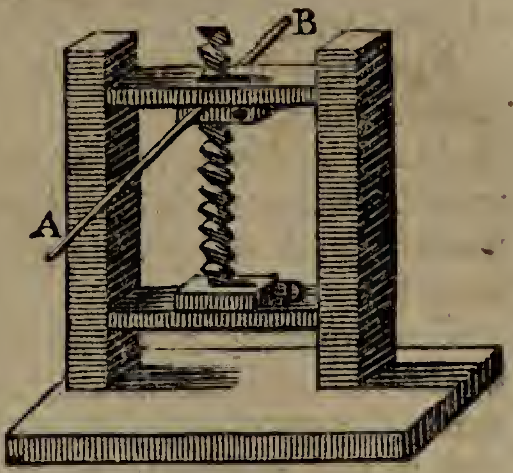
201. *The Force of a Power applied to turn a Screw round, is to the Force with which it presses upward or downward, setting aside the Friction, as the Distance between two Threads, is to the Circumference where the Power is applied.*

The screw being an inclined plane, or half wedge, whose height is the distance between two threads, and its base the circumference of the screw; and the force in the horizontal direction, being to that in the vertical one, as the lines perpendicular to them, namely, as the height of the plane, or distance of the two threads, is to the base of the plane, or circumference of the screw; therefore the power is to the pressure, as the distance of two threads is to that circumference. But, by means of a handle or lever, the gain in power is increased in the proportion of the radius of the screw to the radius of the power, or length of the handle, or as their circumferences. Therefore, finally, the power is to the pressure, as the distance of the threads, is to the circumference described by the power.

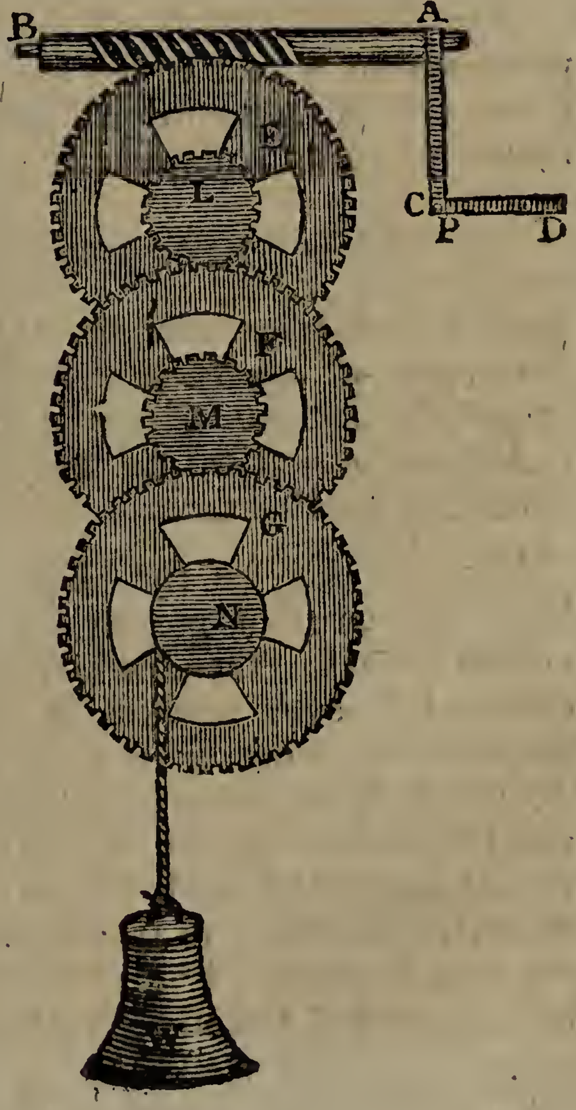
202. *Corol.* When the screw is put in motion; then the power is to the weight which would keep it in equilibrio, as the velocity of the latter is to that of the former; and hence their two momenta are equal, which are produced by multiplying each weight or power by its own velocity. So that this is a general property in all the mechanical powers, namely, that the momentum of a power is equal to that of the weight which would balance it in equilibrio; or that each of them is reciprocally proportional to its velocity.

SCHOLIUM.

203. Hence we can easily compute the force of any machine turned by a screw. Let the annexed figure represent a press driven by a screw, whose threads are each a quarter of an inch asunder; and let the screw be turned by a handle of 4 feet long, from A to B; then, if the natural force of a man, by which he can lift, pull, or draw, be 150 pounds; and it be required to determine with what force the screw will press on the board at D, when the man turns the handle at A and B, with his whole force. Now the diameter AB of the power being 4 feet or 48 inches, its circumference is 48 × 3·1416 or $150\frac{4}{5}$ nearly; and the distance of the threads being $\frac{1}{4}$ of an inch; therefore the power is to the pressure, as 1 to $603\frac{1}{5}$: but the power is equal to 150lb; theref. as 1 : $603\frac{1}{5}$ :: 150 : 90480; and consequently the pressure at D is equal to a weight of 90480 pounds; independent of friction.



204. Again, if the endless screw AB be turned by a handle AC of 20 inches, the threads of the screw being distant half an inch each; and the screw turn a toothed wheel E, whose pinion L turns another wheel F, and the pinion M of this another wheel G, to the pinion or barrel of which is hung a weight W; it is required to determine what weight the man will be able to raise, working at the handle C; supposing the diameters of the wheels to be 18 inches, and those of the pinions and barrel 2 inches; the teeth and pinions being all of a size.



Here

Here $20 \times 3{\cdot}1416 \times 2 = 125{\cdot}664$, is the circumference of the power.

And $125{\cdot}664$ to $\frac{1}{2}$, or $251{\cdot}328$ to 1, is the force of the ſcrew alone.

Alſo, 18 to 2, or 9 to 1, being the proportion of the wheels to the pinions; and as there are three of them, therefore 9^3 to 1, or 729 to 1, is the power gained by the wheels.

Conſequently $251{\cdot}328 \times 729$ to 1, or $183218\frac{1}{9}$ to 1 nearly, is the ratio of the power to the weight, ariſing from the advantage both of the ſcrew and the wheels.

But the power is 150lb; therefore $150 \times 183218\frac{1}{9}$, or 27482716 pounds, is the weight the man can ſuſtain, which is equal to 12269 tons weight.

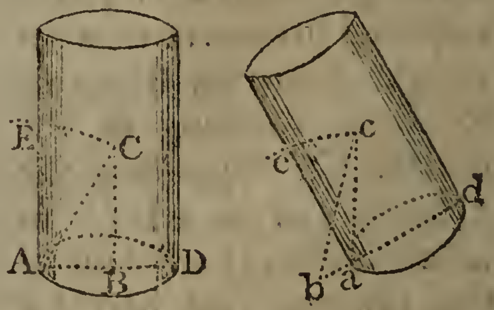
But the power has to overcome, not only the weight, but alſo the friction of the ſcrew, which is very great, in ſome caſes equal to the weight itſelf, ſince it is ſometimes ſufficient to ſuſtain the weight, when the power is taken off.

On the CENTRE of GRAVITY.

205. THE Centre of Gravity of a body, is a certain point within it, on which the body being freely ſuſpended, it will reſt in any poſition; and it will always deſcend to the loweſt place to which it can get, in other poſitions.

PROPOSITION XXXVIII.

206. *If a Perpendicular to the Horizon, from the Centre of Gravity of any Body, fall Within the Baſe of the Body, it will reſt in that Poſition; but if the Perpendicular fall Without the Baſe, the Body will not reſt in that Poſition, but will tumble down.*



For, if CB be the perp. from the centre of gravity C, within the baſe: then the body cannot fall over towards A; becauſe, in turning on the point A, the centre of gravity C would deſcribe an arc which would riſe from C to E; contrary to the nature of that centre, which only reſts when in the loweſt place. For the ſame reaſon, the body will not fall towards D. And therefore it will ſtand in that poſition.

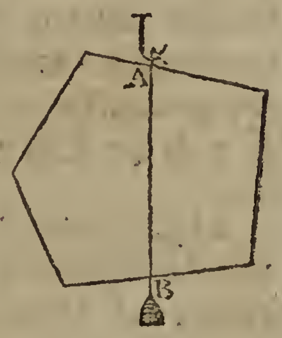
But

But if the perpendicular fall without the baſe, as Cb; then the body will tumble over on that ſide; becauſe, in turning on the point a, the centre c deſcends by deſcribing the deſcending arc ce.

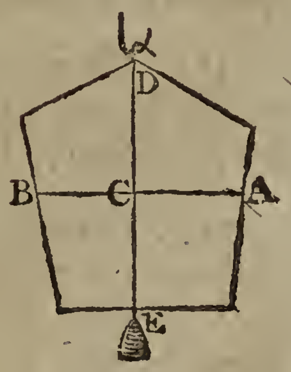
207. *Corol.* 1. If a perpendicular, drawn from the centre of gravity, fall juſt on the extremity of the baſe; the body may ſtand; but any the leaſt force will cauſe it to fall that way. And the nearer the perpendicular is to any ſide, or the narrower the baſe is, the eaſier it will be made to fall, or be puſhed over that way; becauſe the centre of gravity has the leſs height to riſe: which is the reaſon that a globe is made to roll on a ſmooth plane by any the leaſt force. But the nearer the perpendicular is to the middle of the baſe, or the broader the baſe is, the firmer the body ſtands.

208. *Corol.* 2. Hence, if the centre of gravity of a body be ſupported, the whole body is ſupported. And the place of the centre of gravity muſt be accounted the place of the body; for into that point the whole matter of the body may be ſuppoſed to be collected, and therefore all the force alſo with which it endeavours to deſcend.

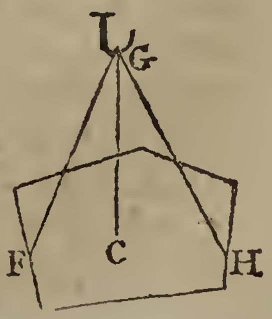
209. *Corol.* 3. From the property which the centre of gravity has, of always deſcending to the loweſt point, is derived an eaſy mechanical method of finding that centre.



Thus, if the body be hung up by any point A, and a plumb line AB be hung by the ſame point, it will paſs through the centre of gravity; becauſe that centre is not in the loweſt point till it fall in the plumb line. Mark the line AB upon it. Then hang the body up by any other point D, with a plumb line DE, which will alſo paſs through the centre of gravity, for the ſame reaſon as before; and therefore that centre muſt be at C where the two plumb lines croſs each other.



210. Or, if the body be ſuſpended by two or more cords GF, GH, &c, then a plumb line from the point G will cut the body in its centre of gravity C.



211. Like-

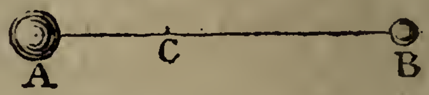
211. Likewiſe, becauſe a body reſts when its centre of gravity is ſupported, but not elſe; we hence derive another eaſy method of finding that centre mechanically. For, if the body be laid on the edge of a priſm, and moved backward and forward till it reſt, or balance itſelf; then is the centre of gravity juſt over the line of the edge. And if the body be then ſhifted into another poſition, and balanced on the edge again, this line will alſo paſs by the centre of gravity; and conſequently the interſection of the two will give the centre itſelf.

PROPOSITION XXXIX.

212. *The Common Centre of Gravity* C *of any two Bodies* A, B, *divides the Line joining their Centres, into two Parts, which are Reciprocally as the Bodies.*

That is, AC : BC :: B : A.

For, if the centre of gravity C be ſupported, the two bodies A and B will be ſupported, and will reſt in equilibrio. But, by the nature of the lever, when two bodies are in equilibrio about a fixed point C, they are reciprocally as their diſtances from that point; therefore A : B :: CB : CA.

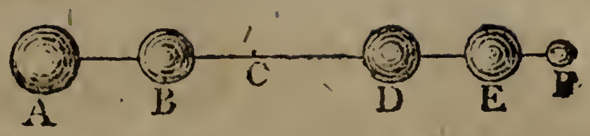


213. *Corol.* 1. Hence, AB : AC :: A + B : B; or, the whole diſtance between the two bodies, is to the diſtance of either of them from the common centre, as the ſum of the bodies is to the other body.

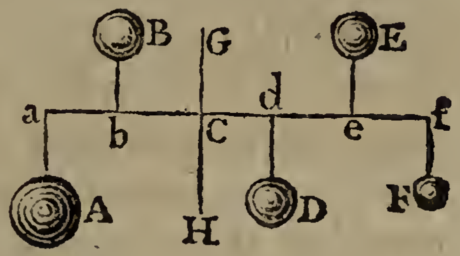
214. *Corol.* 2. Hence alſo, CA . A = CB . B; or the two products are equal, which are made by multiplying each body by its diſtance from the centre of gravity.

215. *Corol.* 3. As the centre C is preſſed with a force equal to both the weights A and B, while the points A and B are each preſſed with the reſpective weights A and B. Therefore, if the two bodies be both united in their common centre C, and only the ends A and B of the line AB be ſupported, each will ſtill bear, or be preſſed by the ſame weights A and B as before. So that, if a weight of 100lb. be laid on a bar at C, ſupported by two men at A and B, diſtant from C, the one 4 feet, and the other 6 feet; then the nearer will bear the weight of 60lb, and the farther only 40lb. weight.

216. *Corol.*



216. *Corol.* 4. Since the effect of any body to turn a lever about the fixed point C, is as that body and its distance from that point; therefore, if C be the common centre of gravity of all the bodies A, B, D, E, F, placed in the straight line AF; then is $CA . A + CB . B = CD . D + CE . E + CF . F$; or, the sum of the products on one side, equal to the sum of the products on the other, made by multiplying each body by its distance from that centre. And if several bodies be in equilibrium upon any straight lever, then the prop is in the centre of gravity.



217. *Corol.* 5. And although the bodies be not situated in a straight line, but scattered about in any promiscuous manner, the same property as in the last corollary still holds true, if perpendiculars to any line whatever af be drawn through the several bodies and their common centre of gravity, namely, that $Ca . A + Cb . B = Cd . D + Ce . E + Cf . F$. For the bodies have the same effect on the line af, to turn it about the point C, whether they are placed at the points a, b, d, e, f, or in any part of the perpendiculars Aa, Bb, Dd, Ee, Ff.

PROPOSITION XL.

218. *If there be three or more Bodies, and, if a Line be drawn from any one Body* D *to the Centre of Gravity of the rest* C; *then the Common Centre of Gravity* E *of all the Bodies, divides the line* CD *into two Parts in* E, *which are Reciprocally Proportional as the Body* D *to the Sum of all the other Bodies.*

That is, $CE : ED :: D : A + B$ &c.

FOR, suppose the bodies A and B to be collected into the common centre of gravity C, and let their sum be called S. Then, by the last prop. $CE : ED :: D : S$ or $A + B$ &c.

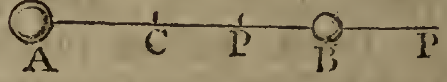
219. *Corol.* Hence we have a method of finding the common centre of gravity of any number of bodies; namely, by first finding the centre of any two of them, then the centre of that centre and a third, and so on for a fourth, or fifth, &c.

PROPOSITION XLI.

220. *If there be taken any Point* P, *in the Line passing through the Centres of two Bodies; then the Sum of the two Products, of each Body multiplied by its Distance from that Point, is equal to the Product of the Sum of the Bodies multiplied by the Distance of their Common Centre of Gravity* C *from the same Point* P.

$$\text{That is, } PA \,.\, A + PB \,.\, B = PC \,.\, \overline{A + B}.$$

For, by the 38th, $CA.A = CB.B$,
that is. $\overline{PA - PC}.A = \overline{PC - PB}.B$;
therefore, by adding,
$PA \,.\, A + PB \,.\, B = PC \,.\, \overline{A + B}$.



221. *Corol.* 1. Hence, the two bodies A and B have the same force to turn the lever about the point P, as if they were both placed in C their common centre of gravity.

Or, if the line, with the bodies, move about the point P; the sum of the momenta of A and B, is equal to the momentum of the sum S or $A + B$ placed at the centre C.

222. *Corol.* 2. The same is also true of any number of bodies whatever, as will appear by cor. 4, prop. 39, namely, $PA \,.\, A + PB \,.\, B + PD \,.\, D \;\&c. = PC \,.\, \overline{A + B + D \;\&c}$, where P is any point whatever in the line AC.

And, by cor. 5, prop. 39, the same thing is true when the bodies are not placed in that line, but any where in the perpendiculars passing through the points A, B, D, &c; namely, $Pa \,.\, A + Pb \,.\, B + Pd \,.\, D \;\&c. = PC \,.\, \overline{A + B + D \;\&c}$.

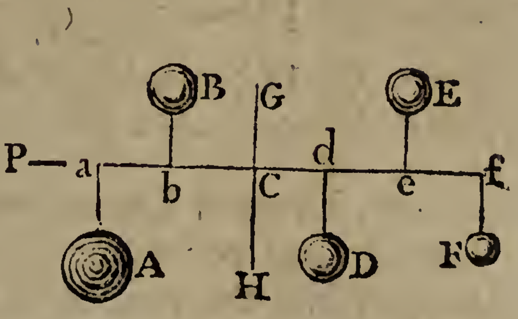
223. *Corol.* 3. And if a plane pass through the point P perpendicular to the line CP; then the distance of the common centre of gravity from that plane, is

$PC = \dfrac{Pa \,.\, A + Pb \,.\, B + Pd \,.\, D \;\&c}{A + B + D \;\&c}$, that is, equal to the sum of all the forces divided by the sum of all the bodies. Or, if A, B, D, &c, be the several particles of one mass or compound body; then the distance of the centre of gravity of the body, below any given point P, is equal to the forces of all the particles divided by the whole mass or body, that is, equal to all the $Pa \,.\, A$, $Pb \,.\, B$, $Pd \,.\, D$, &c, divided by the body or sum of the particles A, B, D, &c.

PRO-

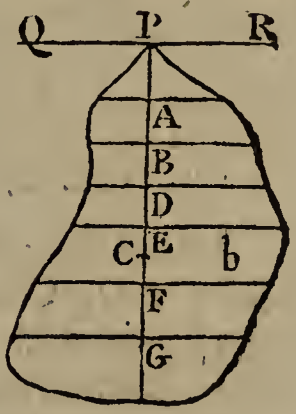
PROPOSITION XLII.

224. *To find the Centre of Gravity of any Body, or of any Syſtem of Bodies.*



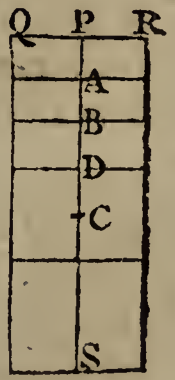
THROUGH any point P draw a plane, and let Pa, Pb, Pd, &c, be the diſtances of the bodies A, B, D, &c, from the plane; then, by the laſt cor. the diſtance of the common centre of gravity from the plane, will be $PC = \frac{Pa.A + Pb.B + Pd.D \,\&c}{A + B + D \,\&c}$

225. Or, if b be any body, and QPR any plane; draw PAB &c, perpendicular to QR, and through A, B, &c, draw innumerable ſections of the body b parallel to the plane QR. Let s denote any of theſe ſections, and $d =$ PA, or PB, &c, its diſtance from the plane QR. Then will the diſtance of the centre of gravity of the body from the plane be $PC = \frac{\text{ſum of all the } ds}{b}$. And if the diſtance be thus found for two interſecting planes, they will give the point in which the centre is placed.



226. But the diſtance from one plane is ſufficient for any regular body, becauſe it is evident that, in ſuch a figure, the centre of gravity is in the axis, or line paſſing through the centres of all the parallel ſections.

Thus, if the figure be a parallelogram, or a cylinder, or any priſm whatever; then the axis or line, or plane PS, which biſects all the ſections parallel to QR, will paſs through the centre of gravity of all thoſe ſections, and conſequently through that of the whole figure C. Then, all the ſections s being equal, and the body $b =$ PS . s, the diſtance of the centre will be $PC = \frac{PA.s + PB.s + \&c}{b} = \frac{PA + PB + PD \,\&c}{b} \times s = \frac{PA + PB + \&c}{PS}$.



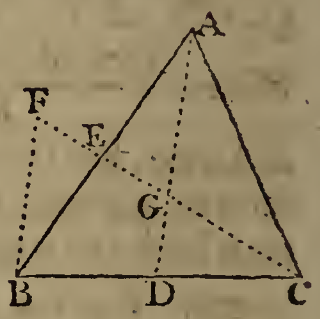
But

But PA + PB + &c, is the ſum of an arithmetical progreſſion, beginning at o, and increaſing to the greateſt term PS, the number of the terms being alſo equal to PS; therefore the ſum PA + BP + &c $= \frac{1}{2}$PS . PS; and conſequently PC $= \frac{\frac{1}{2}PS \,.\, PS}{PS} = \frac{1}{2}$PS; that is, the centre of gravity is in the middle of the axis of any figure whoſe parallel ſections are equal.

227. In other figures, whoſe parallel ſections are not equal, but varying according to ſome general law, it will not be eaſy to find the ſum of all the PA . s, PB . s', PD . s'', &c, except by the general method of Fluxions; which caſe, therefore, will be beſt reſerved till we come to treat of that doctrine. It will be proper, however, to add here ſome examples of another method of finding the centre of gravity of a triangle, or any other right-lined plane figure.

PROPOSITION XLIII.

228. *To find the Centre of Gravity of a Triangle.*



From any two of the angles draw lines AD, CE, to biſect the oppoſite ſides; ſo will their interſection G be the centre of gravity of the triangle.

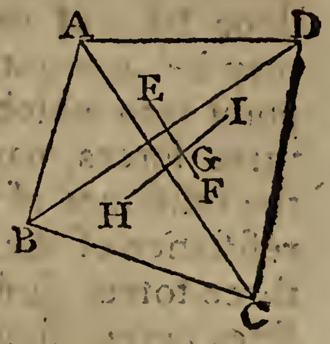
For, becauſe AD biſects BC, it biſects alſo all its parallels, namely, all the parallel ſections of the figure; therefore AD paſſes through the centres of gravity of all the parallel ſections or component parts of the figure; and conſequently the centre of gravity of the whole figure lies in the line AD. For the ſame reaſon, it alſo lies in the line CE. And conſequently it is in their common point of interſection G.

229. *Corol.* The diſtance of the point G, is AG $= \frac{2}{3}$AD, and CG $= \frac{2}{3}$CE; or AG = 2GD, and CG = 2GE.

For, draw BF parallel to AD, and produce CE to meet it in F. Then the triangles AEG, BEF are ſimilar, and alſo equal, becauſe AE = BE; conſequently AG = BF. But the triangles CDG, CBF are alſo equiangular, and CB being = 2CD, therefore BF = 2GD. But BF is alſo = AG; conſequently AG = 2GD or $\frac{2}{3}$AD. In like manner, CG = 2GE or $\frac{2}{3}$CE.

PROPOSITION XLIV.

230. *To find the Centre of Gravity of a Trapezium.*



Divide the trapezium ABCD into two triangles, by the diagonal BD, and find E, F the centres of gravity of these two triangles; then shall the centre of gravity of the trapezium lie in the line EF connecting them. And therefore if EF be divided, in G, in the alternate ratio of the two triangles, namely, EG : GF :: triangle BCD : triangle ABD, then G will be the centre of gravity of the trapezium.

231. Or, having found the two points E, F, if the trapezium be divided into two other triangles BAC, DAC, by the other diagonal AC, and the centres of gravity H and I of these two triangles be also found; then the centre of gravity of the trapezium will also lie in the line HI.

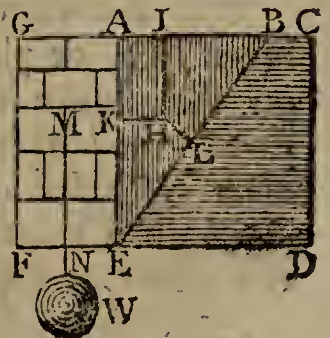
So that, lying in both the lines, EF, HI, it must necessarily lie in their intersection G.

232. And thus we are to proceed for a figure of any greater number of sides, finding the centres of their component triangles and trapeziums, and then finding the common centre of every two of these, till they be all reduced into one only.

Of the use of the place of the centre of gravity, and the nature of forces, the following practical problem is added, to find the force of a bank of earth pressing against a wall, and the force of the wall to support it.

PROPOSITION XLV.

233. *To determine the Force with which a Bank of Earth, or such like, presses against a Wall, and the Dimensions of the Wall necessary to support it.*



Let ACDE be a vertical section of a bank of earth, and suppose that, if it were not supported, a triangular part of it, as ABE, would slide down, leaving it at what is called the natural slope BE; but that, by means of a wall AEFG, it is supported, and kept in its place.—It is required to find the force of ABE, and the dimensions of the wall AEFG.

 Let

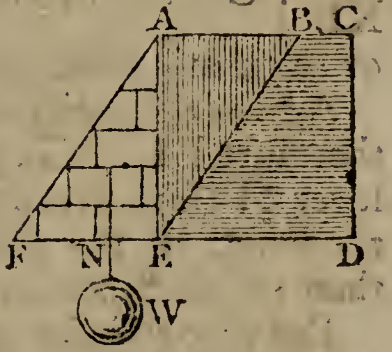
Let H be the centre of gravity of the triangle ABE, from which draw HI perpendicular to the horizontal line AB, HK to the upright face of the wall AE, and HL to the slope face of the earth BE. Now the centre of gravity H may be accounted the place of the triangle ABE, or the point into which it is all collected; and it is sustained by three forces, namely, its weight acting in the direction IH, the resistance of the plane BE in the direction LH, and the resistance of the plane AE in the direction KH; and these three forces, sustaining the body in equilibrio, are as the three lines perpendicular to their directions, namely, as the three lines AB, BE, AE; therefore the weight of the body ABE, is to its pressure against K, as AB is to AE. But $\frac{1}{2}AE \cdot AB$ is the area of the triangle ABE; and if m be the specific gravity of the earth, then $\frac{1}{2}AE \cdot AB \cdot m$ is as its weight. Therefore, as $AB : AE :: \frac{1}{2}AE \cdot AB \cdot m : \frac{1}{2}AE^2 \cdot m$, the force or pressure against K: which therefore is proportional to the square of the altitude AE, whatever be the breadth AB, or the angle of the slope AEB. And the effect of this pressure to overturn the wall, is also as the length of the lever KE or $\frac{2}{3}AE$: consequently its effect is $\frac{1}{2}AE^2 \cdot m \cdot \frac{2}{3}AE$, or $\frac{1}{3}AE^3 \cdot m$. Which must be balanced by the counter resistance of the wall, in order that it may at least be supported.

Now, if M be the centre of gravity of the wall, into which its whole matter may be supposed to be collected, and acting in the direction MNW, its effect will be the same as if a weight W were suspended from the point N of the lever FN. Hence, if A be put for the area of the wall AEFG, and n its specific gravity; then $A \cdot n$ will be equal to the weight W, and $A \cdot n \cdot FN$ its effect on the lever to prevent it from turning about the point F. And as this effort must be equal to that of the triangle of earth, that it may just support it, which was before found equal to $\frac{1}{3}AE^2 \cdot m$; therefore $A \cdot n \cdot FN = \frac{1}{3}AE^3 \cdot m$, in case of an equilibrium.

234. But now, both the breadth of the wall FE, and the lever FN, or place of the centre of gravity M, will depend on the figure of the wall. If the wall be rectangular, or as broad at top as bottom; then $FN = \frac{1}{2}FE$, and the area $A = AE \cdot FE$; consequently the effort of the wall $A \cdot n \cdot FN$ is $= \frac{1}{2}FE^2 \cdot AE \cdot n$; which must be $= \frac{1}{3}AE^3 \cdot m$, the effort of the earth. And the resolution of this equation gives the breadth of the wall $FE = AE\sqrt{\frac{2m}{3n}}$. So that the breadth

breadth of the wall is always proportional to its height, and is always the ſame at the ſame height, whatever the ſlope may be. But the breadth muſt be made a little more than the above value of it, that it may be more than a bare balance to the earth.

235. If the wall be of brick; its ſpecific gravity is about 2000, and that of earth about 1984; namely, m to n as 1984 to 2000: then $\sqrt{\frac{2m}{3n}} = \cdot 813 = \frac{13}{16}$ very nearly; and hence $FE = \frac{13}{16} AE$. That is, whenever a brick rectangular wall is made to ſupport earth, its thickneſs muſt be at leaſt $\frac{13}{16}$ of its height. But if the wall be of ſtone, whoſe ſpecific gravity is about 2520; then $\sqrt{\frac{2m}{3n}} = \cdot 7246 = \frac{8}{11}$ nearly: that is, when the rectangular wall is of ſtone, the breadth muſt be at leaſt $\frac{8}{11}$ of its height.



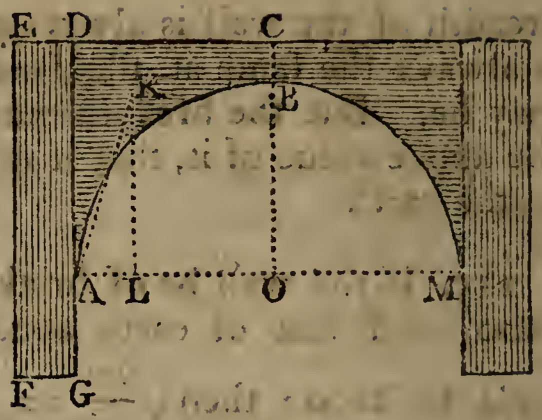
236. But if the figure of the wall be a triangle, the outer ſide tapering to a point at top. Then the lever $FN = \frac{2}{3}FE$, and the area $A = \frac{1}{2}FE \,.\, AE$; conſequently its effort $A \,.\, n \,.\, FN$ is $= \frac{1}{3}FE^2 \,.\, AE \,.\, n$; which being put $= \frac{1}{3}AE^3 \,.\, m$, the equation gives $FE = AE\sqrt{\frac{m}{n}}$ for the breadth of the wall at the bottom, for an equilibrium in this caſe alſo. Where again FE is as AE, and indeed equal to it when the two ſpecific gravities are equal; which is nearly the caſe when this wall is of brick. But when it is of ſtone; then $\sqrt{\frac{m}{n}} = \cdot 8873 = \frac{8}{9}$ nearly: that is, the triangular ſtone wall muſt have its thickneſs at bottom equal to $\frac{8}{9}$ of its height.

And in like manner, for other figures of the wall.

PROPOSITION XLVI.

237. *To determine the Thickneſs of a Pier, neceſſary to ſupport a Given Arch.*

LET



Let ABCD be half the arch, and DEFG the pier. From the centre of gravity K of the arch draw KL perpendicular to the horizon. Then the weight of the arch in direction KL will be to the horizontal push at A in direction LA, as KL to LA.

For the weight of the arch in direction KL, the horizontal push or lateral pressure in direction LA, and the push in direction KA, will be as the three sides KL, LA, KA. So that if A denote the weight or area of the arch; then $\frac{LA}{KL} \cdot A$ will be its force at A in the direction LA; and $\frac{LA}{KL} \cdot GA \cdot A$ its effect on the lever GA to overset the pier, or to turn it about the point F.

Again, the weight or area of the pier, is as EF . FG; and therefore EF . FG . $\frac{1}{2}$FG, or $\frac{1}{2}$EF . FG², is its effect on the lever $\frac{1}{2}$FG, to prevent the pier from being overset; supposing the length of the pier, from point to point, to be no more than the thickness of the arch.

But that the pier and arch be in equilibrio, these two effects must be equal. Therefore we have $\frac{1}{2}EF \cdot FG^2 = \frac{LA}{KL} \cdot GA \cdot A$, and consequently $FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL} \times A}$ is the thickness of the pier as required.

Example 1. Suppose the arc ABM to be a semicircle; and that DC or AC = 45, BC = 6, and GA = 18 feet. Then KL will be found = 40, AL = 15 nearly, and EF = 69; also, the area ABCD or A = 704$\frac{1}{2}$. Therefore $FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL} \cdot A} = \sqrt{\frac{36 \cdot 15}{69 \cdot 40} \cdot 704\frac{1}{2}} = 11\frac{4}{5}$ nearly, which is the thickness of the pier.

Example 2. Suppose, in the segment ABM, AM = 100, OB = 41$\frac{1}{2}$, BC = 6$\frac{1}{2}$, and AG = 10. Then EF = 58, KL = 35, AL = 15 nearly, and ABCD or A = 842. Therefore $FG = \sqrt{\frac{2GA \cdot AL}{EF \cdot KL} \cdot A} = \sqrt{\frac{20 \cdot 15}{58 \cdot 35} \cdot 842} = 11\frac{2}{13}$ nearly, is the thickness of the pier in this case.

On

ON THE CENTRES OF PERCUSSION, OSCILLATION, AND GYRATION.

238. THE CENTRE of PERCUSSION, of a body, or a ſyſtem of bodies, revolving about a point, or axis, is that point, which ſtriking an immoveable object, the whole maſs ſhall not incline to either ſide, but reſt as it were in equilibrio, without acting on the centre of ſuſpenſion.

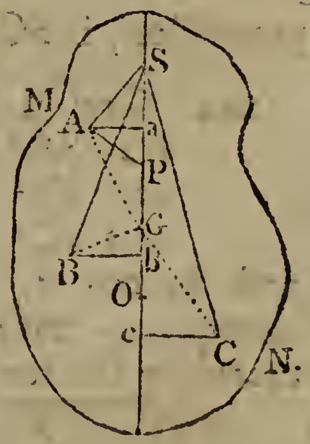
239. The Centre of Oſcillation is that point, in a body vibrating by its gravity, in which if any body be placed, or if the whole maſs be collected, it will perform its vibrations in the ſame time, and with the ſame angular velocity, as the whole body, about the ſame point or axis of ſuſpenſion.

240. The Centre of Gyration, is that point, in which if the whole maſs be collected, the ſame angular velocity will be generated in the ſame time, by a given force acting at any place, as in the body or ſyſtem itſelf.

241. The angular motion of a body, or ſyſtem of bodies, is the motion of a line connecting any point and the centre or axis of motion; and is the ſame in all parts of the ſame revolving body. And in different, unconnected bodies, each revolving about a centre, the angular velocity is as the abſolute velocity directly, and diſtance from the centre inverſely; ſo that, if their abſolute velocities be as their radii or diſtances, the angular velocities will be equal.

PROPOSITION XLVII.

242. *To find the Centre of Percuſſion of a Body, or Syſtem of Bodies.*



LET the body revolve about an axis paſſing through any point S in the line SGO, paſſing through the centres of gravity and percuſſion, G and O. Let MN be the ſection of the body, or the plane in which the axis SGO moves. And conceive all the particles of the body to be reduced to this plane, by perpendiculars let fall from them to the plane: a ſuppoſition which will not affect the centres G, O, nor the angular motion of the body.

Let A be the place of one of the particles, ſo reduced; join

join SA, and draw AP perpendicular to AS, and Aa perpendicular to SGO: then AP will be the direction of A's motion, as it revolves about S; and the whole mass being stopped at O, the body A will urge the point P forward, with a force proportional to its quantity of matter and velocity, or to its matter and distance from the point of suspension S; that is, as A . SA; and the efficacy of this force in a direction perpendicular to SO, at the point P, is as A . Sa, by similar triangles; also, the effect of this force on the lever, to turn it about O, being as the length of the lever, is as $A . Sa . PO = A . Sa . \overline{SO - SP} = A . Sa . SO - A . Sa . SP = A . Sa . SO - A . SA^2$. In like manner, the forces of B and C, to turn the system about O, are as

$$B . Sb . SO - B . SB^2, \text{ and}$$
$$C . Sc . SO - C . SC^2, \&c.$$

But, since the forces on the contrary sides of O destroy one another, by the definition of this force, the sum of the positive parts of these quantities, must be equal to the sum of the negative parts,

that is, $A . Sa . SO + B . Sb . SO + C . Sc . SO \&c = A . SA^2 + B . SB^2 + C . SC^2 \&c$; and

$$\text{hence } SO = \frac{A . SA^2 + B . SB^2 + C . SC^2 \&c}{A . Sa + B . Sb + C . Sc \&c.},$$

which is the distance of the centre of percussion below the axis of motion.

And here it must be observed that, if any of the points a, b, &c, fall on the contrary side of S, the corresponding product A . Sa, or B . Sb, &c, must be made negative.

243. *Corol.* 1. Since, by cor. 3, pr. 40, A + B + C &c, or the body b × the distance of the centre of gravity, SG, is $= A . Sa + B . Sb + C . Sc \&c$, which is the denominator of the value of SO; therefore the distance of the centre of percussion, is $SO = \dfrac{A . SA^2 + B . SB^2 + C . SC^2 \&c}{SG \times \text{body } b}$.

244. *Corol.* 2. Since, by Geometry, theor. 36, 37,

it is $SA^2 = SG^2 + GA^2 - 2SG . Ga$,
and $SB^2 = SG^2 + GB^2 + 2SG . Gb$,
and $SC^2 = SG^2 + GC^2 + 2SG . Gc, \&c$;

and, by cor. 5, pr. 40, the sum of the last terms is nothing, namely, $-2SG . Ga + 2SG . Gb + 2SG . Gc \&c = 0$; therefore the sum of the others, or $A . SA^2 + B . SB^2 \&c$

$= \overline{A + B \&c} . SG^2 + A . GA^2 + B . GB^2 + C . GC^2 \&c$,
or $= b . SG^2 + A . GA^2 + B . GB^2 + C . GC^2 \&c$;

which being substituted in the numerator of the foregoing value of SO, gives

SO =

$$SO = \frac{b \cdot SG^2 + A \cdot GA^2 + B \cdot GB^2 + \&c}{b \cdot SG},$$

$$\text{or } SO = SG + \frac{A \cdot GA^2 + B \cdot GB^2 + C \cdot GC^2 \ \&c}{b \cdot SG}.$$

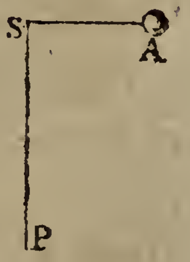
245. *Corol.* 3. Hence the diſtance of the centre of percuſſion, always exceeds the diſtance of the centre of gravity, and the exceſs is always $GO = \frac{A \cdot GA^2 + B \cdot GB^2 \ \&c}{b \cdot SG}$.

246. And hence alſo, $SG \cdot GO = \frac{A \cdot GA^2 + B \cdot GB^2 \ \&c}{\text{the body } b}$;

that is, SG . GO is always the ſame conſtant quantity, wherever the point of ſuſpenſion S is placed; ſince the point G and the bodies A, B, &c, are conſtant. Or GO is always reciprocally as SG, that is, GO is leſs, as SG is greater; and conſequently the point riſes upwards and approaches towards the point G, as the point S is removed to the greater diſtance; and they coincide when SG is infinite. But when S coincides with G, then GO is infinite, or O is at an infinite diſtance.

PROPOSITION XLVIII.

247. *If a Body* A, *at the Diſtance* SA *from an Axis paſſing through* S, *be made to revolve about that Axis by any Force* f *acting at* P *in the Line* SP, *Perpendicular to the Axis of Motion: It is required to determine the Quantity or Matter of another Body* Q, *which being placed at* P, *the Point where the Force acts, it ſhall be accelerated in the Same Manner, as when* A *revolved at the Diſtance* SA; *and conſequently, that the Angular Velocity of* A *and* Q *about* S, *may be the Same in Both Caſes.*



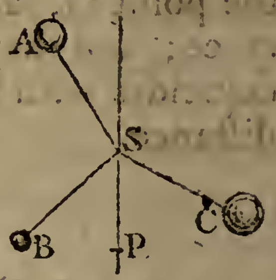
By the nature of the lever, SA : SP :: f : $\frac{SP}{SA} \cdot f$, the effect of the force f, acting at P, on the body at A; that is, the force f acting at P, will have the ſame effect on the body A, as the force $\frac{SP}{SA} \cdot f$, acting directly at the point A. But as ASP revolves altogether about the axis at S, the abſolute velocities of the points A and S, or of the bodies A and Q, will be as the radii SA, SP, of the circles deſcribed by them. Here then we have two bodies A and Q, which being urged directly by the forces f and $\frac{SP}{SA} f$, acquire

velocities

velocities which are as SP and SA. But the motive forces of bodies are as their mafs and velocity: therefore $\frac{SP}{SA}f : f :: A \,.\, SA : Q \,.\, SP$, and $SP^2 : SA^2 :: A : Q = \frac{SA^2}{SP^2}A$, which therefore expreffes the mafs of matter which, being placed at P, would receive the fame angular motion from the action of any force at P, as the body A receives. So that the refiftance of any body A, to a force acting at any point P, is directly as the fquare of its diftance SA from the axis of motion, and reciprocally as the fquare of the diftance SP of the point where the force acts.

248. *Corol.* 1. Hence the force which accelerates the point P, is to the force of gravity, as $\frac{f \,.\, SP^2}{A \,.\, SA^2}$ to 1, or as $f \,.\, SP^2$ to $A \,.\, SA^2$.

249. *Corol.* 2. If any number of bodies A, B, C, be put in motion, about a fixed axis paffing through S, by a force acting at P; the point P will be accelerated in the fame manner, and confequently the whole fyftem will have the fame angular velocity, if, inftead of the bodies A, B, C, placed at the diftances SA, SB, SC, there be fubftituted the bodies $\frac{SA^2}{SP^2}A$, $\frac{SB^2}{SP^2}B$, $\frac{SC^2}{SP^2}C$; thefe being collected into the point P. And hence, the moving force being f, and the matter moved being $\frac{A \,.\, SA^2 + B \,.\, SB^2 + C \,.\, SC^2}{SP^2}$; theref. $\frac{f \,.\, SP^2}{A \,.\, SA^2 + B \,.\, SB^2 + C \,.\, SC^2}$ is the accelerating force; which therefore is to the accelerating force of gravity, as $f \,.\, SP^2$ to $A \,.\, SA^2 + B \,.\, SB^2 + C \,.\, SC^2$.



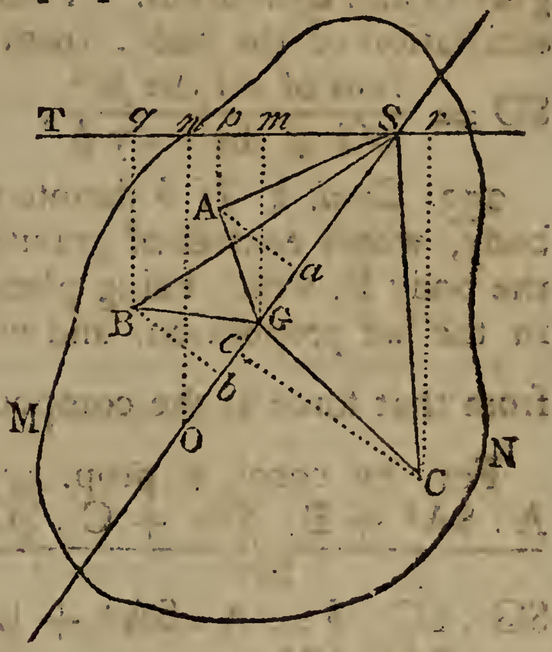
250. *Corol.* 3. The angular velocity of the whole fyftem of bodies, is as $\frac{f \,.\, SP}{A \,.\, SA^2 + B \,.\, SB^2 + C \,.\, SC^2}$. For the abfolute velocity of the point P, is as the accelerating force, or directly as the motive force f, and inverfely as the mafs $\frac{A \,.\, SA^2 \text{ \&c}}{SP^2}$; but the angular velocity is as the abfolute velocity directly, and the radius SP inverfely; and therefore the angular velocity of P, or of the whole fyftem, which is the fame thing, is as $\frac{f \,.\, SP}{A \,.\, SA^2 + B \,.\, SB^2 + C \,.\, SC^2}$.

PRO-

PROPOSITION XLIX.

251. *To determine the Centre of Oscillation of any Compound Mass or Body* MN, *or of any System of Bodies* A, B, C, *&c.*

Let MN be the plane of vibration, to which let all the matter be reduced, by letting fall perpendiculars from every particle, to this plane. Let G be the centre of gravity, and O the centre of oscillation; through the axis S draw SGO, and the horizontal line Sq; then from every particle A, B, C, &c, let fall perpendiculars Aa, Ap, Bb, Bq, Cc, Cr, to these two lines; and join SA, SB, SC; also, draw Gm, On perpendicular to Sq. Now the forces of the weights A, B, C, to turn the body about the axis, are $A . Sp$, $B . Sq$, $- C . Sr$; therefore, by cor. 3, prop. 48, the angular motion generated by all these forces is $\frac{A . Sp + B . Sq - C . Sr}{A.SA^2 + B.SB^2 + C.SC^2}$.



Also, the angular veloc. any particle p, placed in O, generates in the system, by its weight, is $\frac{p . Sn}{p . SO^2}$ or $\frac{Sn}{SO^2}$, or $\frac{Sm}{SG.SO}$; because of the similar triangles SGm, SOn. But, by the problem, the vibrations are performed alike in both cases, and therefore these two expressions must be equal to each other, that is $\frac{Sm}{SG . SO} = \frac{A . Sp + B . Sq - C . Sr}{A . SA^2 + B . SB^2 + C . SC^2}$;

and hence $SO = \frac{Sm}{SG} \times \frac{A . SA^2 + B . SB^2 + C . SC^2}{A . Sp + B . Sq - C . Sr}$.

But, by cor. 2, pr. 41, the sum $A . Sp + B . Sq - C . Sr = (A + B + C) . Sm$; therefore the distance SO =
$$\frac{A . SA^2 + B . SB^2 + C . SC^2}{SG . (A + B + C)} = \frac{A . SA^2 + B . SB^2 + C . SC^2}{A . Sa + B . Sb + C . Sc}$$
by prop. 42, which is the distance of the centre of oscillation O, below the axis of suspension; where any of the products $A . Sa$, $B . Sb$, must be negative, when a, b, &c, lie on the other side of S. So that this is the same expression as that for the distance of the centre of percussion, found in prop. 47.

Hence

Hence it appears, that the centres of percuſſion and of oſcillation, are in the very ſame point. And therefore the properties in all the corollaries there found for the former, are to be here underſtood of the latter.

252. *Corol.* 1. If p be any particle of a body b, and d its diſtance from the axis of motion S; alſo G, O the centres of gravity and oſcillation. Then the diſtance of the centre of oſcillation of the body, from the axis of motion, is - -

$$SO = \frac{\text{ſum of all the } pd^2}{SG \times \text{the body } b}.$$

253. *Corol.* 2. If b denote the matter in any compound body, whoſe centres of gravity and oſcillation are G and O; the body P, which being placed at P, where the force acts as in the laſt propoſition, and which receives the ſame motion from that force as the compound body b, is $P = \frac{SG \cdot SO}{SP^2} \cdot b$.

For, by corol. 2, prop. 47, this body P is = - - -
$\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{SP^2}$. But, by corol. 1, prop. 46, $SG \cdot SO \cdot b = A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2$; therefore $P = \frac{SG \cdot SO}{SP^2} \cdot b$.

SCHOLIUM.

254. By the method of Fluxions, the centre of oſcillation, for a regular body, will be found from cor. 1. But for an irregular one; ſuſpend it at the given point; and hang up alſo a ſimple pendulum of ſuch a length, that, making them both vibrate, they may keep time together. Then the length of the ſimple pendulum, is equal to the diſtance of the centre of oſcillation of the body, below the point of ſuſpenſion.

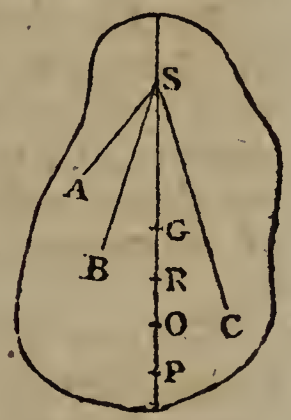
255. Or it will be ſtill better found thus: Suſpend the body very freely by the given point, and make it vibrate in ſmall arcs, counting the number of vibrations it makes in any time, as a minute, by a good ſtop watch; and let that number of vibrations made in a minute be called n: Then ſhall the diſtance of the centre of oſcillation, be $SO = \frac{140850}{nn}$ inches. For, the length of the pendulum vibrating ſeconds, or 60 times in a minute, being $39\frac{1}{8}$ inches; and the lengths of pendulums being reciprocally as the ſquare of the number of vibrations made in the ſame time; therefore - - -
$n^2 : 60^2 :: 39\frac{1}{8} : \frac{60^2 \times 39\frac{1}{8}}{nn} = \frac{140850}{nn}$, the length of the pendulum

pendulum which vibrates n times in a minute, or the diſtance of the centre of oſcillation below the axis of motion.

256. The foregoing determination of the point, into which all the matter of a body being collected, it ſhall oſcillate in the ſame manner as before, only reſpects the caſe in which the body is put in motion by the gravity of its own particles, and the point is the centre of oſcillation: but when the body is put in motion by ſome other extraneous force, inſtead of its gravity, then the point is different from the former, and is called the Centre of Gyration; which is determined in the following manner:

PROPOSITION L.

257. *To determine the Centre of Gyration of a Compound Body or of a Syſtem of Bodies.*



Let R be the centre of gyration, or the point into which all the particles A, B, C, &c, being collected, it ſhall receive the ſame angular motion from a force f acting at P, as the whole ſyſtem receives.

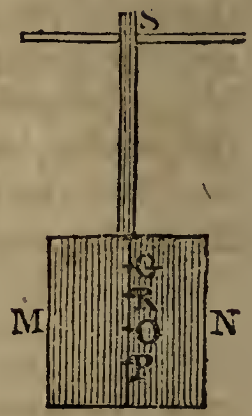
Now, by cor. 3, pr. 47, the angular velocity generated in the ſyſtem by the force f, is as $\frac{f \cdot SP}{A \cdot SA^2 + B \cdot SB^2 \&c}$; and by the ſame, the angular velocity of the ſyſtem placed in R, is $\frac{f \cdot SP}{(A + B + C \&c) \cdot SR^2}$: then, by making theſe two expreſſions equal to each other, the equation gives - - - $SR = \sqrt{\frac{A \cdot SA^2 + B \cdot SB^2 + C \cdot SC^2}{A + B + C}}$, for the diſtance of the centre of gyration below the axis of motion.

258. *Corol.* 1. Becauſe $A.SA^2 + B.SB^2 \&c = SG \cdot SO \cdot b$, where G is the centre of gravity, O the centre of oſcillation, and b the body $A + B + C$ &c; therefore $SR^2 = SG \cdot SO$; that is, the diſtance of the centre of gyration, is a mean proportional between thoſe of gravity and oſcillation.

259. *Corol.* 2. If p denote any particle of a body b, at d diſtance from the axis of motion; then $SR^2 = \frac{\text{ſum of all the } pd^2}{\text{body } b}$.

PROPOSITION LI.

260. *To determine the Velocity with which a Ball moves, which, being shot against a Ballistic Pendulum, causes it to vibrate through a Given Angle.*



The Ballistic Pendulum, is a heavy block of wood MN, suspended vertically by a strong horizontally iron axis at S, to which it is connected by a firm iron stem. This problem is the application of the last proposition, or of prop. 47, and was invented by the very ingenious Mr. Robins, to determine the initial velocities of military projectiles; a circumstance very useful in that science; and it is the best method yet known for determining them with any degree of accuracy.

Let G, R, O be the centres of gravity, gyration, and oscillation, as determined by the foregoing propositions; and let P be the point where the ball strikes the face of the pendulum; the momentum of which, or the product of its weight and velocity, is expressed by the force f, acting at P, in the foregoing propositions. Now,

Put $p =$ the whole weight of the pendulum,
$b =$ the weight of the ball,
$g =$ SG the dist. of the centre of gravity,
$o =$ SO the dist. of the centre of oscillation,
$r =$ SR $= \sqrt{go}$ the dist. of the centre of gyration,
$i =$ SP the distance of the point of impact,
$v =$ the velocity of the ball,
$u =$ the velocity of the point of impact P,
$c =$ chord of the arc described by the point O.

By prop. 49, if the mass p be placed all at R, the pendulum will receive the same motion from the blow in the point P: and as $SP^2 : SR^2 :: p : \frac{SR^2}{SP^2} \cdot p$ or $\frac{r^2}{i^2} p$ or $\frac{go}{ii} p$, (prop. 47), the mass which being placed at P, the pendulum will still receive the same motion as before. Here then are two quantities of matter, namely, b and $\frac{go}{ii} p$, the former moving with the velocity v, and striking the latter at rest; to determine their common velocity u, with which they will jointly proceed forward together after the stroke. In which case,

case, by the law of the impact of non-elastic bodies, we have $\frac{go}{ii}p + b : b :: v : u$, and therefore $v = \frac{bii + gop}{bii}u$ the velocity of the ball in terms of u, the velocity of the point P, and the known dimensions and weights of the bodies.

But now to determine the value of u, we must have recourse to the angle through which the pendulum vibrates; for when the pendulum descends down again to the vertical position, it will have acquired the same velocity with which it began to ascend, and, by the laws of falling bodies, the velocity of the centre of oscillation is such, as a heavy body would acquire by freely falling through the versed sine of the arc described by the same centre O. But the chord of that arc is c, and its radius is o; and, by the nature of the circle, the chord is a mean proportional between the versed sine and diameter, therefore $2o : c :: c : \frac{cc}{2o}$, the versed sine of the arc described by O. Then, by the laws of falling bodies, $\sqrt{16\frac{1}{12}} : \sqrt{\frac{cc}{2o}} :: 32\frac{1}{6} : c\sqrt{\frac{2a}{o}}$, the velocity acquired by the point O in descending through the arc whose chord is c, where $a = 16\frac{1}{12}$ feet: and therefore $o : i :: c\sqrt{\frac{2a}{o}} : \frac{ci}{o}\sqrt{\frac{2a}{o}}$, which is the velocity u, of the point P.

Then, by substituting this value for u, the velocity of the ball, before found, becomes $v = \frac{bii + gop}{bio} \times c\sqrt{\frac{2a}{o}}$. So that the velocity of the ball, is directly as the chord of the arc described by the pendulum in its vibration.

SCHOLIUM.

261. In the foregoing solution, the change in the centre of oscillation is omitted, which is caused by the ball lodging in the point P. But the allowance for that small change, and that of some other small quantities, may be seen in my Tracts, where all the circumstances of this method are treated at full length.

262. For an example in numbers of this method, suppose the weights and dimensions to be as follow: namely,

$p =$

$p = 570$lb,
$b = 18$oz $1\frac{1}{2}$dr
$= 1{\cdot}131$lb,
$g = 78\frac{1}{2}$ inc.
$o = 84\frac{7}{9}$ inc.
$= 7{\cdot}065$ feet
$i = 94\frac{3}{10}$ inc.
$c = 18{\cdot}73$ inc.

Then

$$\frac{bii+gop}{bio} \times c = \frac{1{\cdot}131 \times 94{\cdot}3^2 + 78\frac{1}{2} \times 84\frac{7}{9} \times 570}{1{\cdot}131 \times 94\frac{3}{10} \times 84\frac{7}{9}} \times \frac{18{\cdot}73}{12} = 656{\cdot}56.$$

And $\sqrt{\frac{2a}{o}} = \sqrt{\frac{32\frac{1}{6}}{7{\cdot}065}} = \sqrt{\frac{193}{42{\cdot}39}} = 2{\cdot}1337.$

Therefore $656{\cdot}56 \times 2{\cdot}1337$, or 1401 feet, is the velocity, per ſecond, with which the ball moved when it ſtruck the pendulum.

Of HYDROSTATICS.

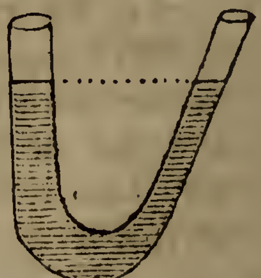
263. Hydrostatics is the ſcience which treats of the preſſure, or weight, and equilibrium of water and other fluids, eſpecially thoſe that are non-elaſtic.

264. A fluid is elaſtic, when it can be reduced into a leſs bulk by compreſſion, and which reſtores itſelf to its former bulk again when the preſſure is removed; as air. And it is non-elaſtic, when it is non compreſſible or expanſible; as water, &c.

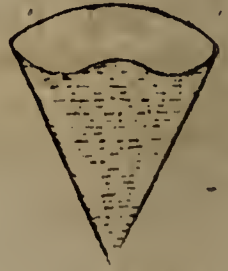
PROPOSITION LII.

265. *If any Part of a Fluid be raiſed higher than the reſt, by any Force, and then left to itſelf; the higher Parts will deſcend to the lower Places, and the Fluid will not reſt, till its Surface be quite even and level.*

For, the parts of a fluid being eaſily moveable every way, the higher parts will deſcend by their ſuperior gravity, and raiſe the lower parts, till the whole come to reſt in a level or horizontal plane.



266. *Corol.* 1. Hence, water which communicates with other water, by means of a cloſe canal or pipe, will ſtand at the ſame height in both places. Like as water in the two legs of a ſyphon.



267. *Corol.* 2. For the ſame reaſon, if a fluid gravitate towards a centre; it will diſpoſe itſelf into a ſpherical figure, the centre of which is the centre of force. Like as the ſea in reſpect of the earth.

PRO-

PROPOSITION LIII.

268. *When a Fluid is at rest in a Veſſel, the Baſe of which is Parallel to the Horizon; Equal Parts of the Baſe are Equally Preſſed by the Fluid.*

For, on every equal part of the baſe there is an equal column of the fluid ſupported by it. And as all the columns are of equal height, by the laſt propoſition, they are of equal weight, and therefore they preſs the baſe equally; that is, equal parts of the baſe ſuſtain an equal preſſure.

269. *Corol.* 1. All parts of the fluid preſs equally at the ſame depth.

For, if a plane parallel to the horizon be conceived to be drawn at that depth; then, the preſſure being the ſame in any part of that plane, by the propoſition, therefore, the parts of the fluid, inſtead of the plane, ſuſtain the ſame preſſure at the ſame depth.

270. *Corol.* 2. The preſſure of the fluid at any depth, is as the depth of the fluid.

For the preſſure is as the weight, and the weight is as the height of the fluid.

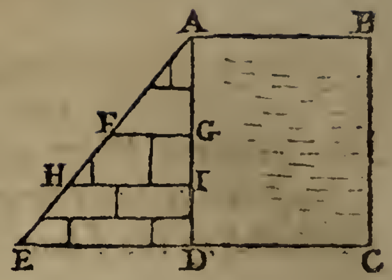
271. *Corol.* 3. The preſſure of the fluid on any horizontal ſurface or plane, is equal to the weight of a column of the fluid, whoſe baſe is equal to that plane, and altitude is its depth below the upper ſurface of the fluid.

PROPOSITION LIV.

272. *When a Fluid is preſſed by its own Weight, or by any other Force; at any Point it preſſes Equally, in All Directions whatever.*

This ariſes from the nature of fluidity, by which it yields to any force in any direction. If it cannot recede from any force applied, it will preſs againſt other parts of the fluid in the direction of that force. And the preſſure in all directions will be the ſame: for if it were leſs in any part, the fluid would move that way, till the preſſure be equal every way.

273. *Corol.* 1. In a veſſel containing a fluid; the preſſure is the ſame againſt the bottom, as againſt the ſides, or even upwards, at the ſame depth.



274. *Corol.* 2. Hence, and from the laſt propoſition, if ABCD be a veſſel of water, and there be taken, in the baſe produced, DE, to repreſent the preſſure at the bottom; joining AE, and drawing any parallels to the baſe, as FG, HI; then ſhall FG repreſent the preſſure at the depth AG, and HI the preſſure at the depth AI, and ſo on; becauſe the parallels - FG, HI, ED, by ſim. triangles, are as the depths AG, AI, AD; which are as the preſſures, by the propoſition.

And hence the ſum of all the FG, HI, &c, or area of the triangle ADE, is as the preſſure againſt all the points G, I, &c, that is, againſt the line AD. But as every point in the line CD is preſſed with a force as DE, and that thence the preſſure on the whole line CD is as the rectangle ED . DC, while that againſt the ſide is as the triangle ADE or $\frac{1}{2}$AD.DE; therefore the preſſure on the horizontal line DC, is to the preſſure againſt the vertical line DA, as DC to $\frac{1}{2}$DA. And hence, if the veſſel be an upright rectangular one, the preſſure on the bottom, or whole weight of the fluid, is to the preſſure againſt one ſide, as the baſe is to half that ſide. And therefore the weight of the fluid is to the preſſure againſt all the four upright ſides, as the baſe is to half the upright ſurface. And the ſame holds true alſo in any upright veſſel, whatever the ſides be, or in a cylindrical veſſel. Or, in the cylinder, the weight of the fluid, is to the preſſure againſt the upright ſurface, as the radius of the baſe is to double the altitude.

Moreover, when the rectangular priſm becomes a cube, it appears that the weight of the fluid on the baſe, is double the preſſure againſt one of the upright ſides, or half the preſſure againſt the whole upright ſurface.

275. *Corol.* 3. The preſſure of a fluid againſt any upright ſurface, as the gate of a ſluice or canal, is equal to half the weight of a column of the fluid whoſe baſe is the ſurface preſſed, and its altitude the ſame as the altitude of that ſurface.

For the preſſure on a horizontal baſe equal to the upright ſurface, is equal to that column; and the preſſure on the upright ſurface is but half that on the baſe, of the ſame area.

So that, if b denote the breadth, and d the depth of ſuch a gate or upright ſurface; then the preſſure againſt it, is equal to the weight of the fluid whoſe magnitude is $\frac{1}{2}bd^2 = \frac{1}{2}AB \cdot AD^2$.

If

If the fluid be water, a cubic foot of which weighs 1000 ounces, or $62\frac{1}{2}$ pounds; and if the depth AD be 12 feet, the breadth AB 20 feet; then the content, or $\frac{1}{2}AB \cdot AD^2$, is 1440 feet; and the pressure is 1440000 ounces, or 90000 pounds, or $40\frac{1}{5}$ tons weight nearly.

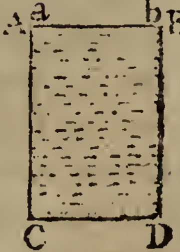
PROPOSITION LV.

276. *The Pressure of a Fluid on a Surface any how immersed in it, either Perpendicular, or Horizontal, or Oblique; is equal to the Weight of a Column of the Fluid, whose Base is equal to the Surface pressed, and its Altitude equal to the Depth of the Centre of Gravity of the Surface pressed below the Top or Surface of the Fluid.*

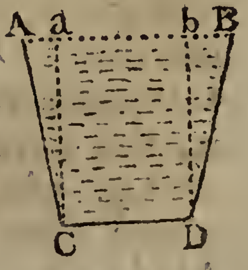
For, conceive the surface pressed to be divided into innumerable sections parallel to the horizon; and let s denote any one of those horizontal sections, also d its distance or depth below the top surface of the fluid. Then, by art. 272, the pressure of the fluid on the section is equal to the weight of ds; consequently the total pressure on the whole surface is equal to all the weights ds. But, if b denote the whole surface pressed, and g the depth of its centre of gravity below the top of the fluid; then, by art. 222 or 225, bg is equal to the sum of all the ds. Consequently the whole pressure of the fluid on the body or surface b, is equal to the weight of the bulk bg of the fluid, that is, of the column whose base is the given surface b, and its height is g the depth of the centre of gravity in the fluid.

PROPOSITION LVI.

277. *The Pressure of a Fluid, on the Base of the Vessel in which it is contained, is as the Base and Perpendicular Altitude; whatever be the Figure of the Vessel that contains it.*



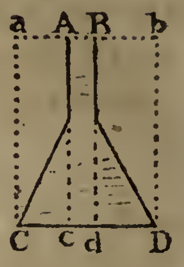
If the sides of the base be upright, so that it be a prism of an uniform width throughout; then the case is evident; for then the base supports the whole fluid, and the pressure is just equal to the weight of the fluid.



But if the vessel be wider at top than bottom; then the bottom sustains, or is pressed by, only the part contained within the upright lines aC, bD; because the parts ACa, BDb are supported by the sides AC, BD; and those parts have no other effect on the part abDC than keeping it in its position, by the lateral pressure against aC and bD, which does not alter its perpendicular pressure down-

 wards.

wards. And thus the preſſure on the bottom is leſs than the weight of the contained fluid.

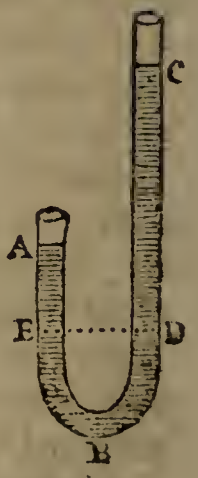


And if the veſſel be wideſt at bottom; then the bottom is ſtill preſſed with a weight which is equal to that of the whole upright column ABDC. For, as the parts of the fluid are in equilibrio, all the parts have an equal preſſure at the ſame depth; ſo that the parts within Cc and dD preſs equally as thoſe in cd, and therefore equally the ſame as if the ſides of the veſſel had gone upright to A and B, the defect of fluid in the parts ACa and BD*b* being exactly compenſated by the downward preſſure or reſiſtance of the ſides aC and bD againſt the contiguous fluid. And thus the preſſure on the baſe may be made to exceed the weight of the contained fluid, in any proportion whatever.

So that, in general, be the veſſels of any figure whatever, regular or irregular, upright or ſloping, or variouſly wide and narrow in different parts, if the baſes and perpendicular altitudes be but equal, the baſes always ſuſtain the ſame preſſure. And as that preſſure, in the regular upright veſſel, is the whole column of the fluid, which is as the baſe and altitude; therefore the preſſure in all figures is in the ſame ratio.

278. *Corol.* 1. Hence, when the heights are equal, the preſſures are as the baſes. And when the baſes are equal, the preſſure is as the heights. But when both the heights and baſes are equal, the preſſures are equal in all, though their contents be ever ſo different.

279. *Corol.* 2. The preſſure on the baſe of any veſſel, is the ſame as on that of a cylinder, of an equal baſe and height.



280. *Corol.* 3. If there be an inverted ſyphon, or bent tube, ABC, containing two different fluids CD, ABD, that balance each other, or reſt in equilibrio; then their heights in the two legs, AE, CD, above the point of meeting, will be reciprocally as their denſities.

For, if they do not meet at the bottom, the part BD balances the part BE, and therefore the part CD balances the part AE; that is, the weight of CD is equal to the weight of AE. And as the ſurface at D is the ſame, where they act againſt each other, therefore AE : CD :: denſity of CD : denſity of AE.

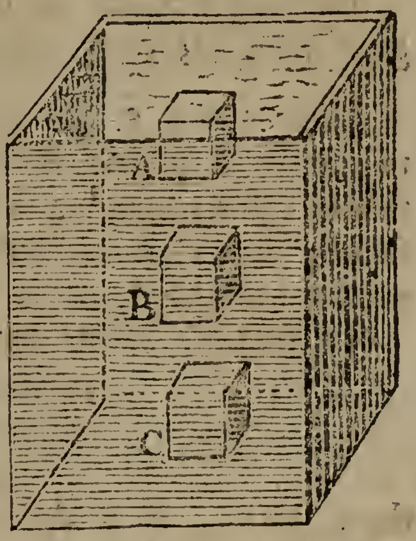
So,

So, if CD be water, and AE quickfilver, which is near 14 times heavier; then CD will be = 14AE; that is, if AE be 1 inch, CD will be 14 inches; if AE be 2 inches, CD will be 28 inches; and fo on.

PROPOSITION LVII.

281. *If a Body be immerfed in a Fluid of the Same Denfity or Specific Gravity; it will reft in any Place where it is put. But a Body of Greater Denfity will Sink; and one of a Lefs Denfity will Afcend to the Top, and Float.*

THE body, being of the fame denfity, or of the fame weight with the like bulk of the fluid, will prefs the fluid under it, juft as much as if its fpace was filled with the fluid itfelf. The preffure then all around it will be the fame as if the fluid were in its place; confequently there is no force, neither upward nor downward, to put the body out of its place. And therefore it will remain wherever it is put.



But if the body be lighter; its preffure downward will be lefs than before, and lefs than the water upward at the fame depth; therefore the greater force will overcome the lefs, and pufh the body upward to A.

And if the body be heavier, the preffure downward will be greater than the fluid at the fame depth; and therefore the greater force will prevail, and carry the body down to the bottom at C.

282. *Corol.* 1. A body immerfed in a fluid, lofes as much weight, as an equal bulk of the fluid weighs. And the fluid gains the fame weight.

Thus, if the body be of equal denfity with the fluid, it lofes all its weight, and fo requires no force but the fluid to fuftain it. If it be heavier, its weight in the water will be only the difference between its own weight and the weight of the fame bulk of water; and it requires a force to fuftain it juft equal to that difference. But if it be lighter, it requires a force equal to the fame difference of weights to keep it from rifing up in the fluid.

283. *Corol.* 2. The weights loft, by immerging the fame body in different fluids, are as the fpecific gravities of the fluids.

fluids. And bodies of equal weight, but different bulk, loſe, in the ſame fluid, weights which are reciprocally as the ſpecific gravities of the bodies, or directly as their bulks.

284. *Corol.* 3. The whole weight of a body which will float in a fluid, is equal to as much of the fluid, as the immerſed part of the body takes up, when it floats.

For the preſſure under the floating body, is juſt the ſame as ſo much of the fluid as is equal to the immerſed part; and therefore the weights are the ſame.

285. *Corol.* 4. Hence the magnitude of the whole body, is to the magnitude of the part immerſed, as the ſpecific gravity of the fluid, is to that of the body.

For, in bodies of equal weight, the denſities, or ſpecific gravities, are reciprocally as their magnitudes.

286. *Corol.* 5. And becauſe, when the weight of a body taken in a fluid, is ſubtracted from its weight out of the fluid, the difference is the weight of an equal bulk of the fluid; this therefore is to its weight in the air, as the ſpecific gravity of the fluid, is to that of the body.

Therefore, if W be the weight of a body in air,
w its weight in water, or any fluid,
S the ſpecific gravity of the body, and
s the ſpecific gravity of the fluid;

then $W - w : W :: s : S$, which proportion will give either of thoſe ſpecific gravities, the one from the other.

Thus $S = \frac{W}{W - w} s$, the ſpecific gravity of the body;

and $s = \frac{W - w}{W} S$, the ſpecific gravity of the fluid.

So that the ſpecific gravities of bodies, are as their weights in the air directly, and their loſs in the ſame fluid inverſely.

287. *Corol.* 6. And hence, for two bodies connected together, or mixed together into one compound, of different ſpecific gravities, we have the following equations, denoting their weights and ſpecific gravities, as below, viz.

H = weight of the heavier body in air,
h = weight of the ſame in water, } S its ſpec. gravity;
L = weight of the lighter body in air,
l = weight of the ſame in water, } s its ſpec. gravity;
C = weight of the compound in air,
c = weight of the ſame in water, } f its ſpec. gravity;
w = the ſpecific gravity of water. Then

1st, $(H - h) S = Hw$,
2d, $(L - l) s = Lw$,
3d, $(C - c) f = Cw$,
4th, $H + L = C$,
5th, $h + l = c$,
6th, $\frac{H}{S} + \frac{L}{s} = \frac{C}{f}$.

From which equations may be found any of the above quantities, in terms of the rest.

Thus, from one of the first three equations, is found the specific gravity of any body, as $s = \frac{Lw}{L - l}$, by dividing the absolute weight of the body by its loss in water, and multiplying by the specific gravity of water.

But if the body L be lighter than water; then l will be negative, and we must divide by $L + l$ instead of $L - l$, and to find l we must have recourse to the compound mass C; and because, from the 4th and 5th equations, $L - l = \overline{C - c} - \overline{H - h}$, therefore $s = \frac{Lw}{(C - c) - (H - h)}$; that is, divide the absolute weight of the light body, by the difference between the losses in water, of the compound and heavier body, and multiply by the specific gravity of water. Or thus, $s = \frac{SfL}{CS - Hf}$, as found from the last equation.

Also, if it were required to find the quantities of two ingredients mixed in a compound, the 4th and 6th equations would give their values as follows, viz.

$$H = \frac{(f - s) S}{(S - s) f} C, \text{ and } L = \frac{(S - f) s}{(S - s) f} C,$$

the quantities of the two ingredients H and L, in the compound C. And so for any other demand.

PROPOSITION LVIII.

To find the Specific Gravity of a Body.

288. CASE 1.—*When the body is heavier than water*; weigh it both in water and out of water, and take the difference, which will be the weight lost in water. Then, by corol. 6, prop. 57, $s = \frac{Bw}{B - b}$, where B is the weight of the body out of water, b its weight in water, s its specific gravity, and w the specific gravity of water. That is,

As the weight lost in water,
Is to the whole or absolute weight,
So is the specific gravity of water,
To the specific gravity of the body.

EXAM-

Example. If a piece of ſtone weigh 10lb, but in water only $6\frac{1}{4}$lb, required its ſpecific gravity, that of water being 1000? Anſ. 3077.

289. Case ii.—*When the body is lighter than water, ſo that it will not ſink;* annex to it a piece of another body, heavier than water, ſo that the maſs compounded of the two may ſink together. Weigh the denſer body and the compound maſs, ſeparately, both in water, and out of it; then find how much each loſes in water, by ſubtracting its weight in water from its weight in air; and ſubtract the leſs of theſe remainders from the greater. Then ſay, by proportion,

As the laſt remainder,
Is to the weight of the light body in air,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

That is, the ſpecific gravity is $s = \frac{Lw}{(C - c) - (H - h)}$, by cor. 6, prop. 57.

Example. Suppoſe a piece of elm weighs 15lb in air; and that a piece of copper, which weighs 18lb in air and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the ſpecific gravity of the elm? Anſ. 600.

290. Case iii.—*For a fluid of any ſort.*—Take a piece of a body of known ſpecific gravity; weigh it both in and out of the fluid, finding the loſs of weight by taking the difference of the two; then ſay,

As the whole or abſolute weight,
Is to the loſs of weight,
So is the ſpecific gravity of the ſolid,
To the ſpecific gravity of the fluid.

That is, the ſpec. grav. $w = \frac{B - b}{B} s$, by cor. 6, pr. 57.

Example. A piece of caſt iron weighed $34\frac{61}{100}$ ounces in a fluid, and 40 ounces out of it; of what ſpecific gravity is that fluid? Anſ. 1000.

PROPOSITION LIX.

291. *To find the Quantities of Two Ingredients in a Given Compound.*

Take the three differences of every pair of the three ſpecific

cific gravities, namely, the ſpecific gravities of the compound and each ingredient; and multiply each ſpecific gravity by the difference of the other two. Then ſay, by proportion,

As the greateſt product,
Is to the whole weight of the compound,
So is each of the other two products,
To the weights of the two ingredients.

That is, the one $H = \frac{(ſ - s)S}{(S - s)ſ}C$; and the other $L = \frac{(S - ſ)s}{(S - s)ſ}C$, by cor. 6, prop. 57.

Example. A compoſition of 112lb being made of tin and copper, whoſe ſpecific gravity is found to be 8784; required the quantity of each ingredient, the ſpecific gravity of tin being 7320, and that of copper 9000?

Anſwer, there is 100lb of copper, and conſequently 12lb of tin. } in the compoſition.

SCHOLIUM.

292. The ſpecific gravities of ſeveral ſorts of matter, as found from experiments, are expreſſed by the numbers annexed to their names in the following Table:

A Table of the Specific Gravities of Bodies.

Platina (pure)	22000	Clay	2160
Fine gold	19640	Brick	2000
Standard gold	18888	Common earth	1984
Quickſilver (pure)	14000	Nitre	1900
Quickſilver (common)	13600	Ivory	1825
Lead	11325	Brimſtone	1810
Fine ſilver	11091	Solid gunpowder	1745
Standard ſilver	10535	Sand	1520
Copper	9000	Coal	1250
Copper halfpence	8915	Box-wood	1030
Gun metal	8784	Sea-water	1030
Caſt braſs	8000	Common water	1000
Steel	7850	Oak	925
Iron	7645	Gunpowder, cloſe ſhaken	937
Caſt iron	7425	Ditto, in a looſe heap	836
Tin	7320	Aſh	800
Clear cryſtal glaſs	3150	Maple	755
Granite	3000	Elm	600
Marble and hard ſtone	2700	Fir	550
Common green glaſs	2600	Charcoal	
Flint	2570	Cork	240
Common ſtone	2520	Air at a mean ſtate	$1\frac{2}{9}$

293. *Note.*

293. *Note.* The ſeveral ſorts of wood are ſuppoſed to be dry. Alſo, as a cubic foot of water weighs juſt 1000 ounces averdupois, the numbers in this table expreſs, not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each, in averdupois ounces; and therefore, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the next two propoſitions.

PROPOSITION LX.

294. *To find the Magnitude of any Body, from its Weight.*

As the tabular ſpecific gravity of the body,
Is to its weight in averdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, reſpectively.

Example 1. Required the content of an irregular block of common ſtone, which weighs 1 cwt, or 112lb?

Anſ. $1228\frac{2016}{2520}$ cubic inches.

Example 2. How many cubic inches of gunpowder are there in 1lb weight? Anſ. 30 cubic inches nearly.

Example 3. How many cubic feet are there in a ton weight of dry oak? Anſ. $38\frac{138}{185}$ cubic feet.

PROPOSITION LXI.

295. *To find the Weight of a Body, from its Magnitude.*

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular ſpecific gravity,
To the weight of the body.

Example 1. Required the weight of a block of marble, whoſe length is 63 feet, and breadth and thickneſs each 12 feet; being the dimenſions of one of the ſtones in the walls of Balbeck?

Anſ. $683\frac{4}{10}$ ton, which is nearly equal to the burthen of an Eaſt-India ſhip.

Example 2. What is the weight of 1 pint, ale meaſure, of gunpowder? Anſ. 19 oz. nearly.

Example 3. What is the weight of a block of dry oak, which meaſures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep or thick? Anſ. $4335\frac{15}{16}$lb.

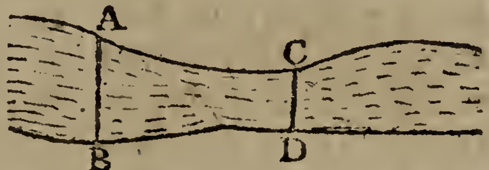
Or

Of HYDRAULICS.

296. Hydraulics is the ſcience which treats of the motion of fluids, and the forces with which they act upon bodies.

PROPOSITION LXII.

297. *If a Fluid run through a Canal or River, or Pipe of various Widths, always filling it; the Velocity of the Fluid in different Parts of it,* AB, CD, *will be Reciprocally as the Transverſe Sections in thoſe Parts.*



That is, veloc. at A : veloc. at C :: $\frac{1}{AB} : \frac{1}{CD}$; or :: CD : AB; where AB and CD denote, not the diameters at A and B, but the areas or ſections, there.

For, as the channel is always equally full, the quantity of water running through AB is equal to the quantity running through CD, in the ſame time; that is, the column through AB is equal to the column through CD, in the ſame time; or AB × length of its column = CD × length of its column; therefore AB : CD :: length of column through CD : length of column through AB. But the uniform velocity of the water, is as the ſpace run over, or length of the columns; therefore AB : CD :: velocity through CD : velocity through AB.

298. *Corol.* Hence, by obſerving the velocity at any place AB, the quantity of water diſcharged in a ſecond, or any other time, will be found, namely, by multiplying the ſection AB by the velocity there.

But if the channel be not a cloſe pipe or tunnel, kept always full, but an open canal or river; then the velocity in all parts of the ſection will not be the ſame, becauſe the velocity towards the bottom and ſides will be diminiſhed by the friction againſt the bed or channel; and therefore a medium among the three ought to be taken. So, if the velocity at the top be - 100 feet per minute,
that at the bottom - 60
and that at the ſides - 50

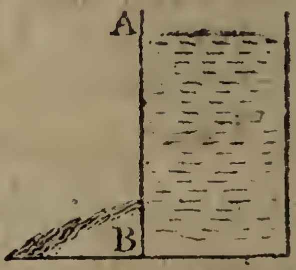
3) 210 ſum;

dividing their ſum by 3 gives 70 for the mean velocity, which is to be multiplied by the ſection, to give the quantity diſcharged in a minute.

PRO-

PROPOSITION LXIII.

299. *The Velocity with which a Fluid runs out by a Hole in the Bottom or Side of a Veſſel, kept always full, is equal to that which is generated by Gravity through the Height of the Water above the Hole; that is, the Velocity of a Heavy Body acquired by falling freely through the Height* AB.



Divide the altitude AB into a great number of very ſmall parts, each being 1, their number a, or $a =$ the altitude AB.

Now, by prop. 54, the preſſure of the fluid againſt the hole B, by which the motion is generated, is equal to the weight of the column of fluid above it, that is the column whoſe height is AB or a, and baſe the area of the hole B. Therefore the preſſure on the hole, or ſmall part of the fluid 1, is to its weight, or the natural force of gravity, as a to 1. But, by art. 28, the velocities generated in the ſame body in any time, are as thoſe forces; and becauſe gravity generates the velocity 2 in deſcending through the ſmall ſpace 1, therefore $1 : a :: 2 : 2a$, the velocity generated by the preſſure of the column of fluid in the ſame time. But $2a$ is alſo, by corol. 1, prop. 6, the velocity generated by gravity in deſcending through a or AB. That is, the velocity of the iſſuing water, is equal to that which is acquired by a body in falling through the height AB.

300. *Corol.* 1. The velocity, and quantity run out, at different depths, are as the ſquare roots of the depths. For the velocity acquired in falling through AB, is as $\sqrt{AB}$.

301. *Corol.* 2. The water ſpouts out with the ſame velocity, whether it be downward, or upward, or ſideways; becauſe the preſſure of fluids is the ſame in all directions, at the ſame depth. And therefore, if the adjutage be turned upward, the jet will aſcend to the height of the ſurface of the water in the veſſel. And this is confirmed by experience, by which it is found that jets really aſcend nearly to the height of the reſervoir, abating a ſmall quantity only, for the friction againſt the ſides, and ſome reſiſtance from the air and from the oblique motion of the water in the hole.

302. *Corol.* 3. The quantity run out in any time, is equal to a column or priſm, whoſe baſe is the area of the hole, and its length the ſpace deſcribed in that time by the velocity acquired by falling through the altitude of the fluid. And the

the quantity is the ſame, whatever be the figure of the orifice, if it is of the ſame area.

Therefore, if h denote the height of the fluid,

a the area of the orifice, and

$g = 16\frac{1}{12}$ feet, or 193 inches;

then $2a\sqrt{gh}$ will be the quantity of water diſcharged in a ſecond of time; or nearly $8\frac{1}{48}a\sqrt{h}$ cubic feet, when a and h are taken in feet.

So, for example, if the height h be 25 inches, and the orifice $a = 1$ ſquare inch; then $2a\sqrt{gh} = 2\sqrt{25 \times 193} = 139$ cubic inches, which is the quantity that would be diſcharged per ſecond.

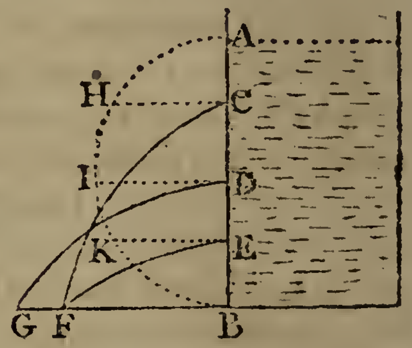
SCHOLIUM.

303. When the orifice is in the ſide of the veſſel, then the velocity is different in the different parts of the hole, being leſs in the upper parts of it than in the lower. However, when the hole is but ſmall, the difference is inconſiderable, and the altitude may be eſtimated from the centre of the hole, to obtain the mean velocity. But when the orifice is pretty large, then the mean velocity is to be more accurately computed by other principles, given in the next propoſition.

304. It is not to be expected that experiments, as to the quantity of water run out, will exactly agree with this theory, both on account of the reſiſtance of the air, the reſiſtance of the water againſt the ſides of the orifice, and the oblique motion of the particles of the water in entering it. For, it is not merely the particles ſituated immediately in the column over the hole, which enter it and iſſue forth, as if that column only were in motion; but alſo particles from all the ſurrounding parts of the fluid, which is in a commotion quite around; and the particles thus entering the hole in all directions, ſtrike againſt each other, and impede one another's motion: from whence it happens, that the real velocity through the orifice, is ſomewhat leſs than that of a ſingle body only, urged with the ſame preſſure of the ſuperincumbent column of the fluid. And experiments on the quantity of water diſcharged through apertures, ſhew that the quantity muſt be diminiſhed, by thoſe cauſes, rather more than the fourth part, when the orifice is ſmall, or ſuch as to make the mean velocity equal to that in a body falling through $\frac{1}{2}$ the height of the fluid above the orifice. Or elſe, that the orifice is not quite full of particles that ſpout out with the whole velocity, aſſigned in the propoſition.

305. Experiments have alſo been made on the extent to which

which the ſpout of water ranges on a horizontal plane, and compared with the theory, by calculating it as a projectile diſcharged with the velocity acquired by deſcending through the height of the fluid. For, when the aperture is in the ſide of the veſſel, the fluid ſpouts out horizontally with a uniform velocity, which, combined with the perpendicular velocity from the action of gravity, cauſes the jet to form the curve of a parabola. Then the diſtances to which the jet will ſpout on the horizontal plane BG, will be as the roots of the rectangles of the ſegments AC . CB, AD . DB, AE . EB. For the ſpaces BF, BG, are as the times and horizontal velocities; but the velocity si as $\sqrt{AC}$, and the time of the fall, which is the ſame as the time of moving, is as $\sqrt{CB}$; therefore the diſtance BF is as $\sqrt{AC \cdot CB}$; and the diſtance BG as $\sqrt{AD \cdot DB}$. And hence, if two holes are made equidiſtant from the top and bottom, they will project the water to the ſame diſtance; for if AC = EB, then the rectangle AC . CB is equal the rectangle AE . EB; which makes EF the ſame for both. Or, if on the diameter AB a ſemicircle be deſcribed; then, becauſe the ſquares of the ordinates CH, DI, EK are equal to the rectangles AC . CB, &c; therefore the diſtances BF, BG are as the ordinates CH, DI. And hence alſo it follows, that the projection from the middle point D will be fartheſt, for DI is the greateſt ordinate.



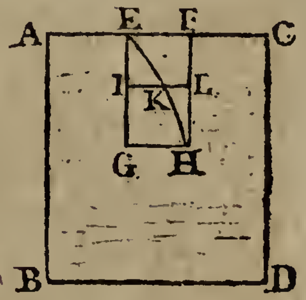
Theſe are the *proportions* of the diſtances: but for the abſolute diſtances, it will be thus. The velocity through any hole C, is ſuch as will carry the water horizontally through a ſpace equal to 2AC in the time of falling through AC: but, after quitting the hole, it deſcribes a parabola, and comes to F in the time a body will fall through CB; and to find this diſtance, ſince the times are as the roots of the ſpaces, therefore $\sqrt{AC} : \sqrt{CB} :: 2AC : 2\sqrt{AC \cdot CB} = 2CH = BF$, the ſpace ranged on the horizontal plane. And the greateſt range BG = 2DI, or 2AD, or equal to AB.

And as theſe ranges anſwer very exactly to the experiments, this confirms the theory, as to the velocity aſſigned.

PRO-

PROPOSITION LXIV.

306. *If a Notch or Slit* EH *in form of a Parallelogram, be cut in the Side of a Vessel, full of Water,* AD; *the Quantity of Water flowing through it, will be* $\frac{2}{3}$ *of the Quantity flowing through an equal Orifice, placed at the whole Depth* EG, *or at the Base* GH, *in the Same Time; it being supposed that the Vessel is always kept full.*



FOR the velocity at GH is to the velocity at IL, as $\sqrt{EG}$ to $\sqrt{EI}$; that is, as GH or IL to IK, the ordinate of a parabola EKH, whose axis is EG. Therefore the sum of the velocities at all the points I, is to as many times the velocity at G, as the sum of all the ordinates IK, to the sum of all the IL's, namely, as the area of the parabola EGH, is to the area EGHF; that is, the quantity running through the notch EH, is to the quantity running through an equal horizontal area placed at GH, as EGHKE to EGHF, or as 2 to 3; the area of a parabola being $\frac{2}{3}$ of its circumscribing parallelogram.

307. *Corol.* 1. The mean velocity of the water in the notch, is equal to $\frac{2}{3}$ of that at GH.

308. *Corol.* 2. The quantity flowing through the hole IGHL, is to that which would flow through an equal orifice placed as low as GH, as the parabolic frustum IGHK, is to the rectangle IGHL. As appears from the demonstration.

OF PNEUMATICS.

309. PNEUMATICS is the science which treats of the properties of air, or elastic fluids.

PROPOSITION LXV.

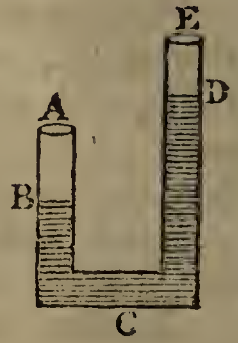
310. *Air is a Heavy Fluid Body; and it Surrounds the Earth, and Gravitates on all Parts of its Surface.*

THESE properties of air are proved by experience.—That it is a fluid, is evident from its easily yielding to any the least force impressed on it, without making a sensible resistance.

But when it is moved briskly, by any means, as by a fan or a pair of bellows; or when any body is moved very briskly through it; in these cases we become sensible of it as a body, by the resistance it makes in such motions, and likewise by its

its impelling or blowing away any light ſubſtances. So that, being capable of reſiſting, or moving other bodies by its impulſe, it muſt itſelf be a body, and be heavy, like all other bodies, in proportion to the matter it contains; and therefore it will preſs on all bodies that are placed under it.

Alſo, as it is a fluid, it will ſpread itſelf all over on the earth; and, like other fluids, it will gravitate and preſs every where on the earth's ſurface.

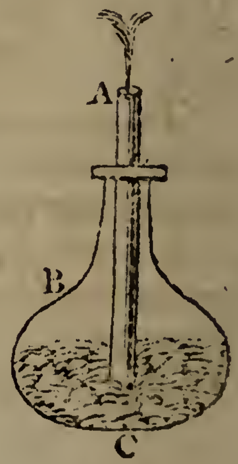


311. The gravity and preſſure of the air is alſo evident from many experiments. Thus, for inſtance, if water, or quickſilver, be poured into the tube ACE, and the air be ſuffered to preſs on it, in both ends of the tube, the fluid will reſt at the ſame height in both legs of the tube: but if the air be drawn out of one end as E, by any means; then the air preſſing on the other end A, will preſs down the fluid in this leg at B, and raiſe it up in the other to D, as much higher than at B, as the preſſure of the air is equal to. By which it appears, not only that the air does really preſs, but alſo what the quantity of that preſſure is equal to. And this is the principle of the barometer.

PROPOSITION LXVI.

312. *The Air is alſo an Elaſtic Fluid, being Condenſible and Expanſible. And the Law it obſerves is this, that its Denſity is proportional to the Force which compreſſes it.*

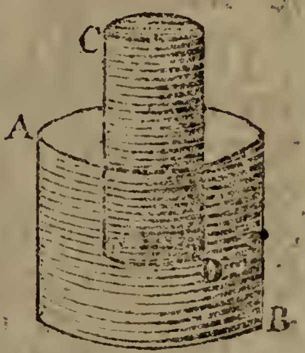
THIS property of the air is proved by many experiments. Thus, if the handle of a ſyringe be puſhed inwards, it will condenſe the incloſed air into leſs ſpace, thereby ſhewing its condenſibility. But the included air, thus condenſed, will be felt to act ſtrongly againſt the hand, reſiſting the force compreſſing it more and more; and, on withdrawing the hand, the handle is puſhed back again to where it was at firſt. Which ſhews that the air is elaſtic.



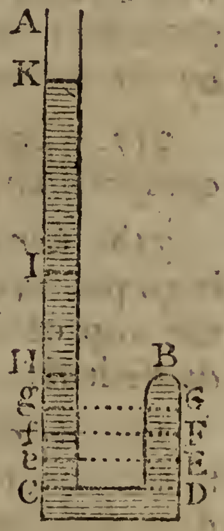
313. Again, fill a ſtrong bottle half full of water, and then inſert a pipe into it, putting its lower end down near to the bottom, and cementing it very cloſe round the mouth of the bottle. Then, if air be ſtrongly injected through the pipe, as by blowing with the mouth or otherwiſe, it will paſs through the water from the lower end, aſcending into the parts before occupied with air at B, and the whole

whole maſs of air become there condenſed, becauſe the water is not compreſſible into a leſs ſpace. But, on removing the force which injected the air at A, the water will begin to riſe from thence in a jet, being puſhed up the pipe by the increaſed elaſticity of the air B, by which it preſſes on the ſurface of the water, and forces it through the pipe, till as much be expelled as there was air forced in; when the air at B will be reduced to the ſame denſity as at firſt, and, the balance being reſtored, the jet will ceaſe.

314. Likewiſe, if into a jar of water AB, be inverted an empty glaſs tumbler CD, or ſuch-like, the mouth downward; the water will enter it, and partly fill it, but not near ſo high as the water in the jar, compreſſing and condenſing the air into a leſs ſpace in the upper parts C, and cauſing the glaſs to make a ſenſible reſiſtance to the hand in puſhing it down. Then, on removing the hand, the elaſticity of the internal condenſed air throws the glaſs up again. All theſe ſhewing that the air is condenſible and elaſtic.

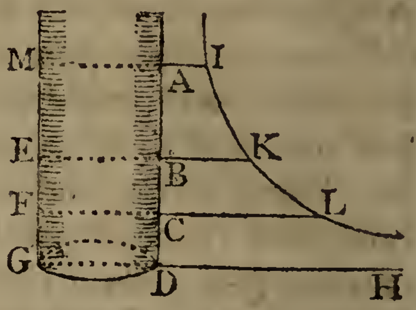


315. Again, to ſhew the rate or proportion of the elaſticity to the condenſation; take a long crooked glaſs tube, equally wide throughout, or at leaſt in the part BD, open at A, but cloſe at the other end B. Pour in a little quickſilver at A, juſt to cover the bottom to the bend at CD, and to ſtop the communication between the external air and the air in BD. Then pour in more quickſilver, and mark the correſponding heights at which it ſtands in the two legs: ſo, when it riſes to H in the open leg AC, let it riſe to E in the cloſe one, reducing its included air from the natural bulk BD to the contracted ſpace BE, by the preſſure of the column He; and when the quickſilver ſtands at I and K, in the open leg, let it riſe to F and G in the other, reducing the air to the reſpective ſpaces BF, BG, by the weights of the columns I*f*, K*g*. Then it is always found, that the condenſations and elaſticities are as the compreſſing weights, or columns, of the quickſilver, and the atmoſphere together. So, if the natural bulk of the air BD be compreſſed into the ſpaces BE, BF, BG, which are $\frac{3}{4}$, $\frac{2}{4}$, $\frac{1}{4}$ of BD, or as the numbers 3, 2, 1; then the atmoſphere, together with the correſponding columns H*e*, I*f*, K*g*, will alſo be



found to be in the ſame proportion reciprocally, viz. as $\frac{1}{3}$, $\frac{1}{2}$, $\frac{1}{1}$, or as the numbers 2, 3, 6. And then $He = \frac{1}{3}A$, $If = A$, and $Kg = 3A$; where A is the weight of the atmoſphere, Which ſhews, that the condenſations are directly as the compreſſing forces. And the elaſticities are in the ſame ratio, ſince the columns in AC are ſuſtained by the elaſticities in BD.

From the foregoing principles may be deduced many uſeful remarks, as in the following corollaries, viz.



316. *Corol.* 1. The ſpace which any quantity of air is confined in, is reciprocally as the force that compreſſes it. So, the forces which confine a quantity of air in the cylindrical ſpaces AG, BG, CG, are reciprocally as the ſame, or reciprocally as the heights AD, BD, CD. And therefore, if to the two perpendicular lines DA, DH, as aſymptotes, the hyperbola IKL be deſcribed, and the ordinates AI, BK, CL be drawn; then the forces which confine the air in the ſpaces AG, BG, CG, will be directly as the correſponding ordinates AI, BK, CL, ſince theſe are reciprocally as the abſciſſes AD, BD, CD, by the nature of the hyperbola.

317. *Corol.* 2. All the air near the earth is in a ſtate of compreſſion, by the weight of the incumbent atmoſphere.

318. *Corol.* 3. The air is denſer near the earth, than in high places; or denſer at the foot of a mountain than at the top of it. And the higher above the earth, the leſs denſe it is.

319. *Corol.* 4. The ſpring or elaſticity of the air, is equal to the weight of the atmoſphere above it; and they will produce the ſame effects; ſince they always ſuſtain and balance each other.

320. *Corol.* 5. If the denſity of the air be increaſed, preſerving the ſame heat or temperature; its ſpring or elaſticity will likewiſe be increaſed, and in the ſame proportion.

321. *Corol.* 6. By the gravity and preſſure of the atmoſphere, on the ſurface of fluids, the fluids are made to riſe in any pipes or veſſels, when the ſpring or preſſure within is decreaſed or taken off.

PRO-

PROPOSITION LXVII.

322. *Heat Increaſes the Elaſticity of the Air, and Cold Diminiſhes it. Or, Heat Expands, and Cold Condenſes the Air.*

This property is alſo proved by experience.

323. THUS, tie a bladder very cloſe with ſome air in it; and lay it before the fire: then as it warms, it will more and more diſtend the bladder, and at laſt burſt it, if the heat be continued, and increaſed high enough. But if the bladder be removed from the fire, as it cools it will contract again, as before. And it was upon this principle, that the firſt air-balloons were made by Montgolfier: for, by heating the air within them, by a fire underneath, the hot air diſtends them to a ſize which occupies a ſpace in the atmoſphere, whoſe weight of common air exceeds that of the balloon.

324. Alſo, if a cup or glaſs, with a little air in it, be inverted into a veſſel of water; and the whole be heated over the fire, or otherwiſe; the air in the top will expand till it fill the glaſs, and expel the water out of it; and part of the air itſelf will follow, by continuing or increaſing the heat.

Many other experiments, to the ſame effect, might be adduced, all proving the properties mentioned in the propoſition.

SCHOLIUM.

325. So that, when the force of the elaſticity of air is conſidered, regard muſt be had to its heat or temperature; the ſame quantity of air being more or leſs elaſtic, as its heat is more or leſs. And it has been found, by experiment, that the elaſticity is increaſed by the 435th part, by each degree of heat, of which there are 180 between the freezing and boiling heat of water.

326. *N. B.* Water expands about the $\frac{3}{20000}$ part, with each degree of heat. (Sir Geo. Shuckburgh, Philoſ. Tranſ. 1777, p. 560, &c.)

Alſo, the

Spec. grav. of air	1·201 or $1\frac{1}{5}$	when the barom. is	29·5,
water	1000	and the thermom. is	55°
mercury	13592	which are their mean heights in this country.	

Or thus,	air	1·222 or $1\frac{2}{9}$	when the barom. is	30,
	water	1000	and thermometer	55.
	mercury	13600		

PROPOSITION LXVIII.

327. *The Weight or Pressure of the Atmosphere, on any Base at the Earth's Surface, is equal to the Weight of a Column of Quicksilver, of the Same Base, and the Height of which is between* 28 *and* 31 *inches.*

This is proved by the barometer, an instrument which measures the pressure of the air, and which is described below. For, at some seasons, and in some places, the air sustains and balances a column of mercury, of about 28 inches: but at other times it balances a column of 29, or 30, or near 31 inches high; seldom in the extremes 28 or 31, but commonly about the means 29 or 30. A variation which depends partly on the different degrees of heat in the air near the surface of the earth, and partly on the commotions and changes in the atmosphere, from winds and other causes, by which it is accumulated in some places, and depressed in others, being thereby rendered denser and heavier, or rarer and lighter; which changes in its state are almost continually happening in any one place. But the medium state is commonly about $29\frac{1}{2}$ or 30 inches.

328. *Corol.* 1. Hence the pressure of the atmosphere on every square inch at the earth's surface, at a medium, is very near 15 pounds averdupois, or rather $14\frac{3}{4}$ pounds.

For, a cubic foot of mercury weighing 13600 ounces nearly, an inch of it will weigh 7·866 or almost 8 ounces, or near half a pound, which is the weight of the atmosphere for every inch of the barometer on a base of a square inch; and therefore 30 inches, or the medium height, weighs very near $14\frac{3}{4}$ pounds.

329. *Corol.* 2. Hence also, the weight or pressure of the atmosphere, is equal to that of a column of water from 32 to 35 feet high, or on a medium 33 or 34 feet high.

For, water and quicksilver are in weight nearly as 1 to 13·6; so that the atmosphere will balance a column of water 13·6 times as high as one of quicksilver; consequently

13·6 times 28 inches = 381 inches, or $31\frac{3}{4}$ feet,
13·6 times 29 inches = 394 inches, or $32\frac{5}{6}$ feet,
13·6 times 30 inches = 408 inches, or 34 feet,
13·6 times 31 inches = 422 inches, or $35\frac{1}{6}$ feet.

And hence a common sucking pump will not raise water higher than about 33 or 34 feet. And a syphon will not run, if the perpendicular height of the top of it be more than about 33 or 34 feet.

330. *Corol.* 3. If the air were of the ſame uniform denſity at every height up to the top of the atmoſphere, as at the ſurface of the earth; its height would be about $5\frac{1}{4}$ miles at a medium.

For, the weights of the ſame bulk of air and water, are nearly as 1·222 to 1000; therefore as 1·222 : 1000 :: $33\frac{3}{4}$ feet : 27600 feet, or $5\frac{1}{4}$ miles nearly. And ſo high the atmoſphere would be, if it were all of uniform denſity, like water. But, inſtead of that, from its expanſive and elaſtic quality, it becomes continually more and more rare, the farther above the earth, in a certain proportion; which will be treated of below, as alſo the method of meaſuring heights by the barometer, which depends on it.

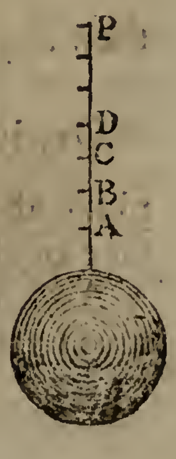
331. *Corol.* 4. From this propoſition and the laſt it follows, that the height is always the ſame, of an uniform atmoſphere above any place, which ſhall be all of the uniform denſity with the air there, and of equal weight or preſſure with the real height of the atmoſphere above that place, whether it be at the ſame place at different times, or at any different places or heights above the earth; and that height is always about $5\frac{1}{4}$ miles, or 27600 feet, as above found. For, as the denſity varies in exact proportion to the weight of the column, therefore it requires a column of the ſame height in all caſes, to make the reſpective weights or preſſures. Thus, if W and w be the weights of atmoſphere above any places, D and d their denſities, and H and h the heights of the uniform columns, of the ſame denſities and weights; Then $H \times D = W$,

and $h \times d = w$;

therefore $\frac{W}{D}$ or H is equal to $\frac{w}{d}$ or h. The temperature being the ſame.

PROPOSITION LXIX.

332. *The Denſity of the Atmoſphere, at different Heights above the Earth, Decreaſes in ſuch Sort, that when the Heights increaſe in Arithmetical Progreſſion, the Denſities decreaſe in Geometrical Progreſſion.*



Let the indefinite perpendicular line AP, erected on the earth, be conceived to be divided into a great number of very ſmall equal parts A, B, C, D, &c, forming ſo many thin ſtrata of air in the atmoſphere, all of different denſity, gradually decreaſing from the greateſt at A: then the denſity of the ſeveral ſtrata A, B, C, D, &c, will be in geometrical progreſſion decreaſing.

For,

For, as the ſtrata A, B, C, &c, are all of equal thickneſs, the quantity of matter in each of them, is as the denſity there; but the denſity in any one, being as the compreſſing force, is as the weight or quantity of all the matter from that place upward to the top of the atmoſphere; therefore the quantity of matter in each ſtratum, is alſo as the whole quantity from that place upward. Now, if from the whole weight at any place as B, the weight or quantity in the ſtratum B be ſubtracted, the remainder is the weight at the next ſtratum C: that is, from each weight ſubtracting a part which is proportional to itſelf, leaves the next weight; or, which is the ſame thing, from each denſity ſubtracting a part which is always proportional to itſelf, leaves the next denſity. But when any quantities are continually diminiſhed by parts which are proportional to themſelves, the remainders form a ſeries of continued proportionals; conſequently theſe denſities are in geometrical progreſſion.

Thus, if the firſt denſity be D, and from each be taken its nth part; then there will remain its $\frac{n-1}{n}$ part, or the $\frac{m}{n}$ part, putting m for $n-1$; and therefore the ſeries of denſities will be D, $\frac{m}{n}$D, $\frac{m^2}{n^2}$D, $\frac{m^3}{n^3}$D, $\frac{m^4}{n^4}$D, &c, the common ratio of the ſeries being that of n to m.

SCHOLIUM.

333. Becauſe the terms of an arithmetical ſeries, are proportional to the logarithms of the terms of a geometrical ſeries; therefore different altitudes above the earth's ſurface, are as the logarithms of the denſities, or of the weights of air, at thoſe altitudes.

So that, if D denote the denſity at the altitude A,
and d - the denſity at the altitude a;
then A being as the log. of D, and a as the log. of d,
the dif. of alt. $A - a$ will be as the log. D — log. d or log. $\frac{D}{d}$.

And if $A = 0$, or D the denſity at the ſurface of the earth; then any alt. above the ſurface a, is as the log. of $\frac{D}{d}$.

Or, in general, the log. of $\frac{D}{d}$ is as the altitude of the one place above the other, whether the lower place be at the ſurface of the earth, or any where elſe.

And from this property is derived the method of determining the heights of mountains and other eminences, by the

the barometer, which is an inſtrument that meaſures the preſſure or denſity of the air at any place. For, by taking, with this inſtrument, the preſſure or denſity, at the foot of a hill for inſtance, and again at the top of it, the difference of the logarithms of theſe two preſſures, or the logarithm of their quotient, will be as the difference of altitude, or as the height of the hill; ſuppoſing the temperatures of the air to be the ſame at both places, and the gravity of air not altered by the different diſtances from the earth's centre.

334. But as this formula expreſſes only the relations between different altitudes, with reſpect to their denſities, recourſe muſt be had to ſome experiment, to obtain the real altitude which correſponds to any given denſity, or the denſity which correſponds to a given altitude. And there are various experiments by which this may be done. The firſt, and moſt natural, is that which reſults from the known ſpecific gravity of air, with reſpect to the whole preſſure of the atmoſphere on the ſurface of the earth. Now, as the altitude a is always as log. $\frac{D}{d}$; aſſume h ſo that $a = h \times \log. \frac{D}{d}$, where h will be of one conſtant value for all altitudes; and to determine that value, let a caſe be taken in which we know the altitude a correſponding to a known denſity d; as for inſtance, take $a = 1$ foot, or 1 inch, or ſome ſuch ſmall altitude; then, becauſe the denſity D may be meaſured by the preſſure of the atmoſphere, or the uniform column of 27600 feet, when the temperature is 55°; therefore 27600 feet will denote the denſity D at the lower place, and 27599 the leſs denſity d at 1 foot above it; conſequently $1 = h \times \log. \frac{27600}{27599}$; which, by the nature of logarithms, is nearly $= h \times \frac{\cdot 43429448}{27600}$ $= \frac{h}{63551}$ nearly; and hence $h = 63551$ feet; which gives, for any altitude in general, this theorem, viz. $a = 63551 \times \log. \frac{D}{d}$, or $= 63551 \times \log. \frac{M}{m}$ feet, or $10592 \times \log. \frac{M}{m}$ fathoms; where M is the column of mercury which is equal to the preſſure or weight of the atmoſphere at the bottom, and m that at the top of the altitude a; and where M and m may be taken in any meaſure, either feet, or inches, &c.

335. Note, that this formula is adapted to the mean temperature of the air 55°: But, for every degree of tempera-

ture

ture different from this, in the medium between the temperatures at the top and bottom of the altitude a, that altitude will vary by its 435th part; which muſt be added when that medium exceeds 55°, otherwiſe ſubtracted.

336. Note alſo, that a column of 30 inches of mercury varies its length by about the $\frac{1}{330}$ part of an inch for every degree of heat, or rather $\frac{1}{9900}$ of the whole volume.

337. But the formula may be rendered much more convenient for uſe, by reducing the factor 10592 to 10000, by changing the temperature proportionally from 55°: thus, as the diff. 592 is the 18th part of the whole factor 10592; and as 18 is the 24th part of 435; therefore the correſponding change of temperature is 24°, which reduces the 55° to 31°. So that the formula is, $a = 10000 \times \log. \frac{M}{m}$ fathoms, when the temperature is 31 degrees; and for every degree above that, the reſult is to be increaſed by ſo many times its 435th part.

338. *Exam.* 1. To find the height of a hill when the preſſure of the atmoſphere is equal to 29·68 inches of mercury at the bottom, and 25·28 at the top; the mean temperature being 50°? Anſ. 4378 feet, or 730 fathoms.

339. *Exam.* 2. To find the height of a hill when the atmoſphere weighs 29·45 inches of mercury at the bottom, and 26·82 at the top, the mean temperature being 33°? Anſ. 2385 feet, or $397\frac{1}{2}$ fathoms.

340. *Exam.* 3. At what altitude is the denſity of the atmoſphere only the 4th part of what it is at the earth's ſurface? Anſ. 6020 fathoms.

By the weight and preſſure of the atmoſphere, the effect and operations of pneumatic engines may be accounted for, and explained; ſuch as ſyphons, pumps, barometers, &c; of which it may not be improper here to give a brief deſcription.

OF THE SIPHON.

341. THE Siphon, or Syphon, is any bent tube, having its two legs either of equal or of unequal length.


A

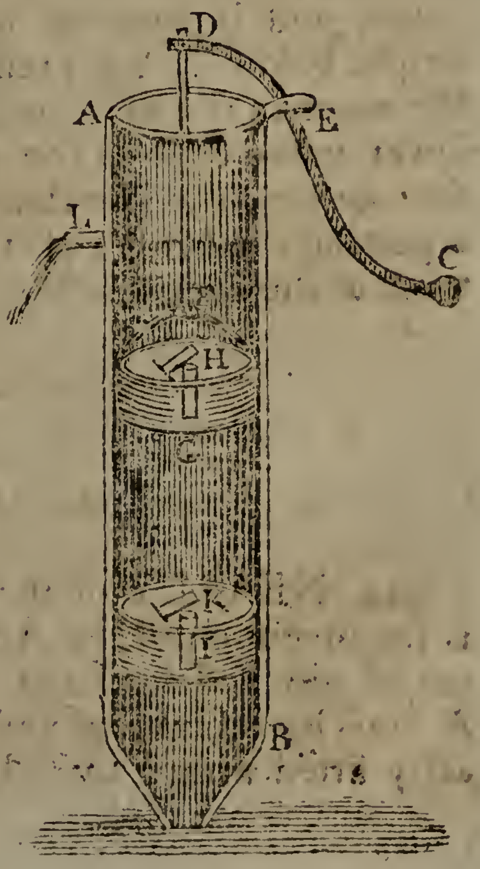
If it be filled with water, and then inverted, with the two open ends downward, and held level in that poſition; the water will remain ſuſpended

ed in it, if the two legs be equal. For the atmofphere will prefs equally on the furface of the water in each end, and fupport them, if they are not more than 34 feet high; and the legs being equal, the water in them is an exact counter-poife by their equal weights; fo that the one has no power to move more than the other; and they are both fupported by the atmofphere.

But if now the fyphon be a little inclined to one fide, fo that the orifice of one end be lower than that of the other; or if the legs be of unequal length, which is the fame thing; then the equilibrium is deftroyed, and the water will all defcend out by the lower end, and rife up in the higher. For, the air preffing equally, but the two ends weighing unequally, a motion muft commence where the power is greateft, and fo continue till all the water has run out by the lower end. And if the fhorter leg be immerfed into a veffel of water, and the fyphon be fet a running as above, it will continue to run till all the water be exhaufted out of the veffel, or at leaft as low as that end of the fyphon. Or, it may be fet a running without filling the fyphon as above, by only inverting it, with its fhorter leg into the veffel of water; then, with the mouth applied to the lower orifice A, fuck the air out; and the water will prefently follow, being forced up into the fyphon by the preffure of the air on the water in the veffel.

Of the PUMP.

342. THERE are three forts of Pumps; the Sucking, the Lifting, and the Forcing Pump. By the former, water can be raifed only to about 34 feet, viz. by the preffure of the atmofphere; but by the others, to any height; but then they require more apparatus and power.



The annexed figure reprefents a common fucking pump. AB is the barrel of the pump, being a hollow cylinder, made of metal, and fmooth within, or of wood for very common purpofes. CD is the handle, moveable about the pin D, by moving the end C up and down. DF an

an iron rod turning about a pin D, which connects it to the end of the handle. This rod is fixed to the piston, bucket, or sucker, FG, by which this is moved up and down within the barrel, which it must fit very tight and close, that no air or water may pass between the piston and the sides of the barrel; and for this purpose it is commonly armed with leather. The piston is made hollow, or it has a perforation through it, the orifice of which is covered by a valve H opening upwards. I is a plug firmly fixed in the lower part of the barrel, also perforated, and covered by a valve K opening upwards.

343. When the pump is first to be worked, and the water is below the plug I; raise the end C of the handle, then the piston descending, compresses the air in HI, which by its spring shuts fast the valve K, and pushes up the valve H, and so enters into the barrel above the piston. Then putting the end C of the handle down again, raises the piston or sucker, which lifts up with it the column of air above it, the external atmosphere by its pressure keeping the valve H shut: the air in the barrel being thus exhausted, or rarefied, is no longer a counterpoise to that which presses on the surface of the water in the well, this is forced up the pipe, and through the valve K, into the barrel of the pump. Then pushing the piston down again into this water, now in the barrel, its weight shuts the lower valve K, and its resistance forces up the valve of the piston, and enters the upper part of the barrel, above the piston. Then, the bucket being raised, lifts up with it the water which had passed above its valve, and it runs out by the cock L; and taking off the weight below it, the pressure of the external atmosphere on the water in the well again forces it up through the pipe and lower valve close to the piston, all the way as it ascends, thus keeping the barrel always full of water. And thus, by repeating the strokes of the piston, a continued discharge is made at the cock L.

Of the AIR-PUMP.

344. NEARLY on the same principles as the water-pump, is the invention of the Air-pump, by which the air is drawn out of any vessel, like as water is drawn out by the former. A brass barrel is bored and polished truly cylindrical, and exactly fitted with a turned piston, so that no air can pass by the

the ſides of it, and furniſhed with a proper valve opening upward. Then, by lifting up the piſton, the air in the cloſe veſſel below it follows the piſton, and fills the barrel; and being thus diffuſed through a larger ſpace than before, when it occupied the veſſel or receiver only, but not the barrel, it is made rarer than it was before, in proportion as the capacity of the barrel and receiver together, exceeds the receiver alone. Another ſtroke of the piſton exhauſts another barrel of this now rarer air, which again rarefies it in the ſame proportion as before. And ſo on, for any number of ſtrokes of the piſton, ſtill exhauſting in the ſame geometrical progreſſion, of which the ratio is that which the capacity of the receiver and barrel together exceeds the receiver, till this is exhauſted to any propoſed degree, or as far as the nature of the machine is capable of performing; which happens when the elaſticity of the included air is ſo far diminiſhed, by rarefying, that it is too feeble to puſh up the valve of the piſton, and eſcape.

345. From the nature of this exhauſting, in geometrical progreſſion, we may eaſily find how much the air in the receiver is rarefied by any number of ſtrokes of the piſton; or what number of ſuch ſtrokes is neceſſary, to exhauſt the receiver to any given degree. Thus, if the capacity of the receiver and barrel together, be to that of the receiver alone, as c to r, and 1 denote the natural denſity of the air at firſt; then

$c : r :: 1 : \frac{r}{c}$, the denſity after 1 ſtroke of the piſton,

$c : r :: \frac{r}{c} : \frac{r^2}{c^2}$, the denſity after 2 ſtrokes,

$c : r :: \frac{r^2}{c^2} : \frac{r^3}{c^3}$, the denſity after 3 ſtrokes,

&c, and $\frac{r^n}{c^n}$, the denſity after n ſtrokes.

So, if the barrel be equal to $\frac{1}{4}$ of the receiver; then $c : r :: 5 : 4$; and $\frac{4^n}{5^n} = 0{\cdot}8^n$ is $= d$ the denſity after n turns. And if n be 20, then $0{\cdot}8^{20} = {\cdot}0115$ is the denſity of the included air after 20 ſtrokes of the piſton; which being the $86\frac{7}{10}$ part of 1, or the firſt denſity, it follows that the air is $86\frac{7}{10}$ times rarefied by the 20 ſtrokes.

346. Or, if it were required to find the number of ſtrokes neceſſary to rarefy the air any number of times; be-

cause $\frac{r^n}{c^n}$ is = the proposed density d; therefore, taking the logarithms, $n \times \log. \frac{r}{c} = \log. d$, and $n = \frac{\log. d}{l. r - l. c}$, the number of strokes required. So, if r be $\frac{4}{5}$ of c, and it be required to rarefy the air 100 times: then $d = \frac{1}{100}$ or ·01; and hence $n = \frac{\log. 100}{l. 5 - l. 4} = 20\frac{3}{5}$ nearly. So that in $20\frac{3}{5}$ strokes the air will be rarefied 100 times.

OF THE DIVING BELL AND CONDENSING MACHINE.

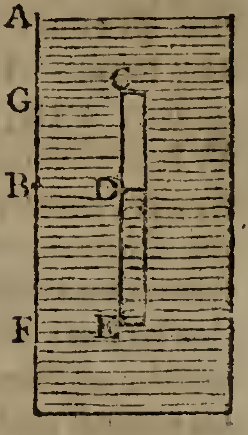
347. ON the same principles too, depend the operations and effect of the Condensing Engine, by which air may be condensed to any degree, instead of rarefied as in the air-pump. And, like as the air-pump rarefies the air, by extracting always one barrel of air after another; so, by this other machine, the air is condensed by throwing in or adding always one barrel of air after another; which it is evident may be done by only turning the valves of the piston and barrel, that is, making them to open the contrary way, and working the piston in the same manner; so that, as they both open upward, or outward, in the air-pump, or rarefier, they will both open downward, or inward, in the condenser.

348. And on the same principles, namely, of the compression and elasticity of the air, depends the use of the Diving Bell, which is a large vessel, in which a person descends to the bottom of the sea, the open end of the vessel being downward; only, in this case the air is not condensed by forcing more of it into the same space, as in the condensing engine; but by compressing the same quantity of air into a less space in the bell, by increasing always the force which compresses it.

349. If a vessel of any sort be inverted into water, and pushed or let down to any depth in it; then by the pressure of the water some of it will ascend into the vessel, but not so high as the water without, and will compress the air into less space, according to the difference between the heights of the internal and external water; and the density and elastic force of the air will be increased in the same proportion, as its space in the vessel is diminished.

So,

So, if the tube CE be inverted, and puſhed down into water, till the external water exceed the internal, by the height AB, and the air of the tube be reduced to the ſpace CD; then that air is preſſed both by a column of water of the height AB, and by the whole atmoſphere which preſſes on the upper ſurface of the water; conſequently the ſpace CD is to the whole ſpace CE, as the weight of the atmoſphere, is to the weights both of the atmoſphere and the column of water AB. So that, if AB be about 34 feet, which is equal to the force of the atmoſphere, then CD will be equal to $\frac{1}{2}$CE; but if AB be double of that, or 68 feet, then CD will be $\frac{1}{3}$CE; and ſo on. And hence, by knowing the depth AF, to which the veſſel is ſunk, we can eaſily find the point D, to which the water will riſe within it at any time. For let the weight of the atmoſphere at that time be equal to that of 34 feet of water; alſo, let the depth AF be 20 feet, and the length of the tube CE 4 feet: then, putting the height of the internal water DE $= x$,



it is - $34 + AB : 34 :: CE : CD$,
that is, $34 + AF - DE : 34 :: CE : CE - DE$,
or - $54 - x : 34 :: 4 : 4 - x$;

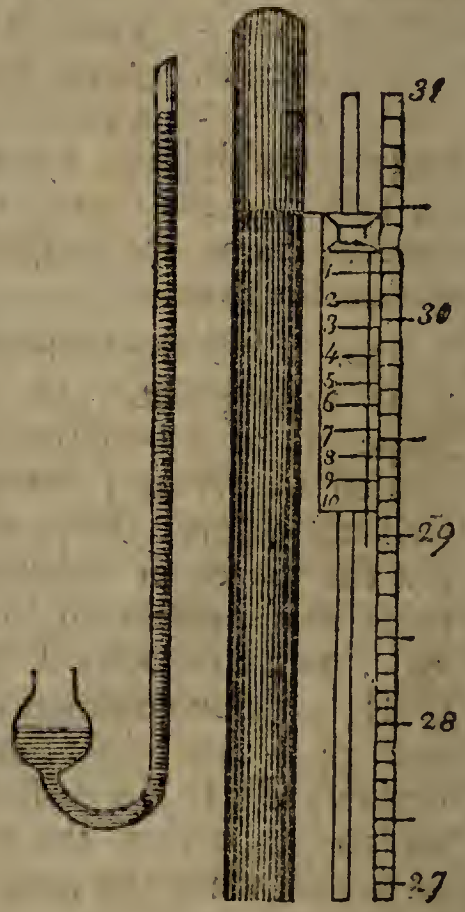
hence, multiplying extremes and means, $216 - 58x + x^2 = 136$, and the root is $x = 1{\cdot}414$ of a foot, or 17 inches nearly; being the height DE to which the water will riſe within the tube.

350. But if the veſſel be not equally wide throughout, but of any other ſhape, as of a bell-like form, ſuch as is uſed in diving; then the altitudes will not obſerve the proportion above, but the ſpaces or bulks only, will reſpect that proportion, namely, $34 + AB : 34 ::$ capacity CKL : capacity CHI, if it be common or freſh water; and $33 + AB : 33 ::$ capacity CKL : capacity CHI, if it be ſea-water. From which proportion, the height DE may be found, when the nature or ſhape of the veſſel or bell CKL is known.

Of

OF THE BAROMETER.

351. THE BAROMETER is an inſtrument for meaſuring the preſſure of the atmoſphere, and elaſticity of the air, at any time. It is commonly made of a glaſs tube, of near 3 feet long, cloſe at one end, and filled with mercury. When the tube is full, by ſtopping the open end with the finger, then inverting the tube, and immerſing that end with the finger into a baſon of quickſilver, on removing the finger from the orifice, the quickſilver in the tube will deſcend into the baſon, till what remains in the tube be of the ſame weight with a column of the atmoſphere, which is commonly between 28 and 31 inches of quickſilver; and leaving an entire vacuum in the upper end of the tube above the mercury. For, as the upper end of the tube is quite void of air, there is no preſſure downwards but from the column of quickſilver, and therefore that will be an exact balance to the counter preſſure of the whole column of atmoſphere, acting on the orifice of the tube by the quickſilver in the baſon. The upper three inches of the tube, namely, from 28 to 31 inches, have a ſcale attached to them, divided into inches, tenths, and hundredths, for meaſuring the length of the column at all times, by obſerving which diviſion of the ſcale the top of the quickſilver is oppoſite to; as it aſcends and deſcends within theſe limits, according to the ſtate of the atmoſphere.



So that the weight of the quickſilver in the tube, above that in the baſon, is at all times equal to the weight or preſſure of the column of atmoſphere above it, and of the ſame baſe with the tube; and hence the weight of it may at all times be computed; being nearly at the rate of half a pound averdupois for every inch of quickſilver in the tube, on every ſquare inch of baſe; or more exactly, it is $\frac{59}{120}$ of a pound on the ſquare inch, for every inch in the altitude of the quickſilver: for the cubic inch of quickſilver weighs juſt $\frac{59}{120}$ lb, or nearly $\frac{1}{2}$ a pound, in the mean temperature of 55°

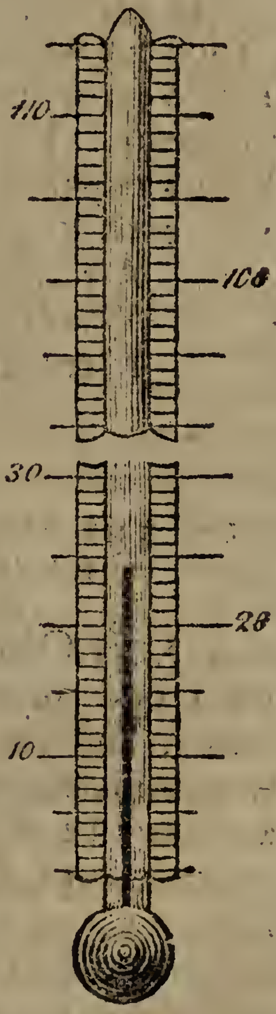
of

of heat. And consequently, when the barometer stands at 30 inches, or $2\frac{1}{2}$ feet high, which is the medium or standard height, the whole pressure of the atmosphere is equal to $14\frac{1}{6}$ pounds, on every square inch of the base. And so in proportion for other heights.

Of the THERMOMETER.

352. THE Thermometer is an instrument for measuring the temperature of the air, as to heat and cold.

It is found by experience, that all bodies expand by heat, and contract by cold; and hence the degrees of expansion become the measure of the degrees of heat. Fluids are more convenient for this purpose, than solids: and quicksilver is now most commonly used for it.



A very fine glass tube, having a pretty large hollow ball at the bottom, is filled about half way up with quicksilver: the whole being then heated very hot till the quicksilver rise quite to the top, the top is then hermetically sealed, so as perfectly to exclude all communication with the outward air. Then, in cooling, the quicksilver contracts, and consequently its surface descends in the tube, till it come to a certain point, correspondent to the temperature or heat of the air. And when the weather becomes warmer, the quicksilver expands, and its surface rises in the tube; and again contracts and descends when the weather becomes cooler. So that, by placing a scale of any divisions against the side of the tube, it will show the degrees of heat, by the expansion and contraction of the quicksilver in the tube; observing at what division of the scale the top of the quicksilver stands. And the method of preparing the scale, as used in England, is thus:—Bring the thermometer into a temperature of just freezing, by immersing the ball in water just freezing, or in ice just thawing, and mark the scale where the mercury then stands, for the point of freezing. Next, immerge it in boiling water; and the quicksilver will rise to a certain height in the tube; which mark also on the scale, for

for the boiling point, or the heat of boiling water. Then the diſtance between theſe two points is divided into 180 equal diviſions, or degrees; and the like equal degrees are alſo continued to any extent below the freezing point, and above the boiling point. The diviſions are then numbered as follows, namely, at the freezing point is ſet the number 32, and conſequently 212 at the boiling point; and all the other numbers in their order.

This diviſion of the ſcale, is commonly called Fahrenheit's. According to this diviſion, 55 is at the mean temperature of the air in this country; and it is in this temperature, and in an atmoſphere which ſuſtains a column of 30 inches of quickſilver in the barometer, that all meaſures and ſpecific gravities are taken, unleſs when otherwiſe mentioned; and in this temperature and preſſure, the relative weights or ſpecific gravities, of air, water, and quickſilver, are as

$$\left.\begin{array}{r} 1\frac{2}{9} \text{ for air,} \\ 1000 \text{ for water,} \\ 13600 \text{ for mercury;} \end{array}\right\{ \begin{array}{l} \text{and theſe alſo are the weights of a cubic} \\ \text{foot of each, in averdupois ounces, in} \\ \text{that ſtate of the barometer and ther-} \end{array}$$

mometer. For other ſtates of the thermometer, each of theſe bodies expands or contracts according to the following rate, with each degree of heat; viz.

Air about - $\frac{1}{435}$ part of its bulk,
Water about $\frac{1}{6666}$ part of its bulk,
Mercury about $\frac{1}{9600}$ part of its bulk.

ON THE MEASUREMENT OF ALTITUDES BY THE BAROMETER AND THERMOMETER.

353. FROM the principles laid down in the Scholium to prop. 69, concerning the meaſuring of altitudes by the barometer, and the foregoing deſcriptions of the barometer and thermometer, we may now collect together the precepts for the practice of ſuch meaſurements, which are as follow:

Firſt, Obſerve the height of the barometer at the bottom of any height, or depth, intended to be meaſured; with the temperature of the quickſilver by means of a thermometer attached to the barometer, and alſo the temperature of the air in the ſhade by a detached thermometer.

Secondly. Let the ſame thing be done alſo at the top of the ſaid height or depth, and at the ſame time, or as near the ſame time as may be. And let thoſe altitudes of barometer be reduced to the ſame temperature, if it be thought neceſſary, by correcting either the one or the other, that is, augment

augment the height of the mercury in the colder temperature; or diminiſh that in the warmer, by its $\frac{1}{9600}$ part for every degree of difference of the two.

Third, Take the difference of the common logarithms of the two heights of the barometer, corrected as above if neceſſary, cutting off three figures next the right hand for decimals, the reſt being fathoms in whole numbers.

Fourth, Correct the number laſt found for the difference of temperature of the air, as follows: Take half the ſum of the two temperatures, for the mean one; and for every degree which this differs from the temperature 31°, take ſo many times the $\frac{1}{435}$ part of the fathoms above found, and add them if the mean temperature be above 31°, but ſubtract them if the mean temperature be below 31°; and the ſum or difference will be the true altitude in fathoms; or, being multiplied by 6, it will be the altitude in feet.

354. *Example* 1. Let the ſtate of the barometers and thermometers be as follows; to find the altitude, viz.

Barom.	Thermom. attach.	Thermom. detach.	
			Anſ. the alt. is
Lower 29·68	57	57	$727\frac{3}{10}$ fath.
Upper 25·28	43	42	

355. *Exam.* 2. To find the altitude, when the ſtate of the barometers and thermometers is as follows, viz.

Barom.	Thermom. attach.	Thermom. detach.	
			Anſ. the alt. is
Lower 29·45	38	31	$408\frac{1}{8}$ fath.
Upper 26·82	41	35	

ON THE RESISTANCE OF FLUIDS, WITH THEIR FORCES AND ACTION ON BODIES.

PROPOSITION LXX.

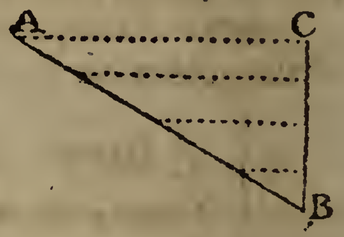
356. *If any Body move through a Fluid at Reſt, or the Fluid move againſt the Body at Reſt; the Force or Reſiſtance of the Fluid againſt the Body, will be as the Square of the Velocity and the Denſity of the Fluid. That is*, $R \propto dv^2$.

For, the force or reſiſtance is as the quantity of matter or particles ſtruck, and the velocity with which they are ſtruck. But the quantity or number of particles ſtruck, in any time, are

are as the velocity and the denſity of the fluid. Therefore the reſiſtance, or force of the fluid, is as the denſity and ſquare of the velocity.

357. *Corol.* 1. The reſiſtance to any plane, is alſo more or leſs, as the plane is greater or leſs; and therefore the reſiſtance on any plane, is as the area of the plane a, the denſity of the medium, and the ſquare of the velocity. That is, $R \propto adv^2$.

358. *Corol.* 2. If the motion be not perpendicular, but oblique to the plane, or to the face of the body; then the reſiſtance, in the direction of motion, will be diminiſhed in the triplicate ratio of radius to the ſine of the angle of inclination of the plane to the direction of motion, or as the cube of radius to the cube of the ſine of that angle. So that $R \propto adv^2s^3$, putting $1 =$ radius, and $s =$ ſine of the angle of inclination CAB.



For, if AB be the plane, AC the direction of motion, and BC perpendicular to AC; then no more particles meet the plane than what meet the perpendicular BC, and therefore their number is diminiſhed as AB to BC, or as 1 to s. But the force of each particle, ſtriking the plane obliquely in the direction CA, is alſo diminiſhed as AB to BC, or as 1 to s; therefore the reſiſtance, which is perpendicular to the face of the plane by art. 52, is as 1^2 to s^2. But again, this reſiſtance in the direction perpendicular to the face of the plane, is to that in the direction AC, by art. 51, as AB to BC, or as 1 to s. Conſequently, on all theſe accounts, the reſiſtance to the plane when moving perpendicular to its face, is to that when moving obliquely, as 1^3 to s^3, or 1 to s^3. That is, the reſiſtance in the direction of the motion, is diminiſhed as 1 to s^3, or in the triplicate ratio of radius to the ſine of inclination.

PROPOSITION LXXI.

359. *The Real Reſiſtance to a Plane, by a Fluid acting in a Direction Perpendicular to its Face, is equal to the Weight of a Column of the Fluid, whoſe Baſe is the Plane, and Altitude equal to that which is due to the Velocity of the Motion, or through which a Heavy Body muſt fall to acquire that Velocity.*

THE reſiſtance to the plane moving through a fluid, is the ſame

ſame as the force of the fluid in motion with the ſame velocity, on the plane at reſt. But the force of the fluid in motion, is equal to the weight or preſſure which generates that motion; and this is equal to the weight or preſſure of a column of the fluid, whoſe baſe is the area of the plane, and its altitude that which is due to the velocity.

360. *Corol.* 1. If a denote the area of the plane, v the velocity, n the denſity or ſpecific gravity of the fluid, and $g = 16\frac{1}{12}$ feet, or 193 inches. Then, the altitude due to the velocity v being $\frac{v^2}{4g}$, therefore $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$ will be the whole reſiſtance, or motive force R.

361. *Corol.* 2. If the direction of motion be not perpendicular to the face of the plane, but oblique to it, in any angle, whoſe ſine is s. Then the reſiſtance to the plane will be $\frac{anv^2s^3}{4g}$.

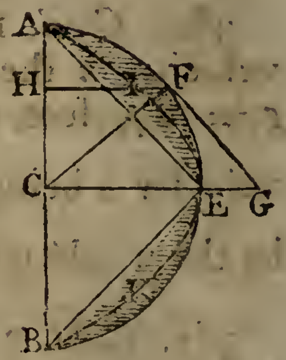
362. *Corol.* 3. Alſo, if w denote the weight of the body, whoſe plane face a is reſiſted by the abſolute force R; then the retarding force f, or $\frac{R}{w}$, will be $\frac{anv^2s^3}{4gw}$.

363. *Corol.* 4. And if the body be a cylinder, whoſe face or end is a, and radius r, moving in the direction of its axis: becauſe then $s = 1$, and $a = pr^2$, where $p = 3{\cdot}1416$; then $\frac{pnv^2r^2}{4g}$ will be the reſiſting force R, and $\frac{pnv^2r^2}{4gw}$ the retarding force f.

364. *Corol.* 5. This is the value of the reſiſtance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face an elliptic ſection, or a conical ſurface, or any other figure every where equally inclined to the axis, or direction of motion, the ſine of inclination being s: then, the number of particles of the fluid ſtriking the face being ſtill the ſame, but the force of each, oppoſed to the direction of motion, diminiſhed in the duplicate ratio of radius to the ſine of inclination, the reſiſting force R would be $\frac{pnr^2v^2s^2}{4g}$.

PROPOSITION LXXII.

365. *The Resistance to a Sphere moving through a Fluid, is but Half the Resistance to its Great Circle, or to the End of a Cylinder of the Same Diameter, moving with an equal Velocity.*



LET AFEB be half the sphere, moving in the direction CEG. Describe the paraboloid AIEKB on the same base. Let any particle of the medium meet the semicircle in F, to which draw the tangent FG, the radius FC, and the ordinate FIH. Then the force of any particle on the surface at F, is to its force on the base at H, as the square of the sine of the angle G, or its equal the angle FCH, to the square of radius, that is, as HF^2 to CF^2. Therefore the force of all the particles, or the whole fluid, on the whole surface, is to its force on the circle of the base, as all the HF^2 to as many times CF^2. But CF^2 is $= CA^2 = AC . CB$, and $HF^2 = AH . HB$ by the nature of the circle; also, $AH . HB : AC . CB :: HI : CE$ by the nature of the parabola; consequently the force on the spherical surface, is to the force on its circular base, as all the HI's to as many CE's; that is, as the content of the paraboloid to the ontent of its circumscribed cylinder, namely, as 1 to 2.

366. *Corol.* Hence, the resistance to the sphere is $R = \frac{pnv^2r^2}{8g}$, being the half of that of a cylinder of the same diameter. For example, a 9lb iron ball, whose diameter is 4 inches, when moving through the air with a velocity of 1600 feet per second, would meet a resistance which is equal to a weight of $132\frac{4}{7}$lb, over and above the pressure of the atmosphere, for want of the counterpoize behind the ball.

PRACTICAL EXERCISES IN MENSURATION.

QUEST. 1. WHAT difference is there between a floor 28 feet long by 20 broad, and two others, each of half the dimensions; and what do all three come to at 45s. per square, or 100 square feet?

Ans. dif. 280 sq. feet. Amount 18 guineas.

QUEST. 2.

QUEST. 2. An elm plank is 14 feet 3 inches long, and I would have juſt a ſquare yard ſlit off it; at what diſtance from the edge muſt the line be ſtruck? Anſ. $7\frac{1}{1}\frac{1}{9}$ inches.

QUEST. 3. A ceiling contains 114 yards 6 feet of plaſtering, and the room 28 feet broad; what is the length of it?
Anſ. $36\frac{6}{7}$ feet.

QUEST. 4. A common joiſt is 7 inches deep, and $2\frac{1}{2}$ thick; but I want a ſcantling juſt as big again, that ſhall be 3 inches thick; what will the other dimenſion be?
Anſ. $11\frac{2}{3}$ inches.

QUEST. 5. A wooden ciſtern coſt me 3s. 2d. painting within, at 6d. per yard; the length of it was 102 inches, and the depth 21 inches; what was the width?
Anſ. $27\frac{1}{4}$ inches.

QUEST. 6. If my court-yard be 47 feet 9 inches ſquare, and I have laid a foot-path with Purbeck ſtone, of 4 feet wide, along one ſide of it; what will paving the reſt with flints come to, at 6d. per ſquare yard? Anſ. 5l. 16s. $0\frac{1}{2}$d.

QUEST. 7. A ladder, 40 feet long, may be ſo planted, that it ſhall reach a window 33 feet from the ground on one ſide of the ſtreet; and, by only turning it over, without moving the foot out of its place, it will do the ſame by a window 21 feet high on the other ſide: what is the breadth of the ſtreet? Anſ. 56 feet $7\frac{3}{4}$ inches.

QUEST. 8. The paving of a triangular court, at 18d. per foot, came to 100l; the longeſt of the three ſides was 88 feet; required the ſum of the other two equal ſides?
Anſ. 106·85 feet.

QUEST. 9. There are two columns in the ruins of Perſepolis left ſtanding upright; the one is 64 feet above the plain, and the other 50: in a ſtraight line between theſe ſtands an ancient ſmall ſtatue, the head of which is 97 feet from the ſummit of the higher, and 86 feet from the top of the lower column, the baſe of which meaſures juſt 76 feet to the centre of the figure's baſe. Required the diſtance between the tops of the two columns? Anſ. 157 feet nearly.

QUEST. 10. The perambulator, or ſurveying wheel, is ſo contrived, as to turn juſt twice in the length of 1 pole, or $16\frac{1}{2}$ feet; required the diameter? Anſ. 2·626 feet.

QUEST. 11. In turning a one-horſe chaiſe within a ring of a certain diameter, it was obſerved, that the outer wheel made two turns, while the inner made but one: the wheels were

were both 4 feet high; and, supposing them fixed at the statutable distance of 5 feet asunder on the axletree, what was the circumference of the track described by the outer wheel? Ans. 62·832 feet.

QUEST. 12. What is the side of that equilateral triangle, whose area cost as much paving at 8d. a foot, as the pallisading the three sides did at a guinea a yard?

Ans. 72·746 feet.

QUEST. 13. In the trapezium ABCD, are given, AB = 13, BC = $31\frac{1}{3}$, CD = 24, and DA = 18, also B a right angle; required the area? Ans. 410·122.

QUEST. 14. A roof, which is 24 feet 8 inches by 14 feet 6 inches, is to be covered with lead at 8lb. per square foot: what will it come to at 18s. per cwt. Ans. 22l. 19s. $10\frac{1}{4}$d.

QUEST. 15. Having a rectangular marble slab, 58 inches by 27, I would have a square foot cut off parallel to the shorter edge; I would then have the like quantity divided from the remainder parallel to the longer side; and this alternately repeated, till there shall not be the quantity of a foot left: what will be the dimensions of the remaining piece? Ans. 20·7 inches by 6·086.

QUEST. 16. Given two sides of an obtuse-angled triangle, which are 20 and 40 poles; required the third side, that the triangle may contain just an acre of land?

Ans. 58·876 or 23·099.

QUEST. 17. The end wall of a house is 24 feet 6 inches in breadth, and 40 feet to the eaves; $\frac{1}{3}$ of which is 2 bricks thick, $\frac{1}{3}$ more is $1\frac{1}{2}$ brick thick, and the rest 1 brick thick. Now the triangular gable rises 38 courses of bricks, 4 of which usually make a foot in depth, and this is but $4\frac{1}{2}$ inches, or half a brick thick: what will this piece of work come to at 5l. 10s. per statute rod? Ans. 20l. 11s. $7\frac{1}{2}$d.

QUEST. 18. How many bricks will it take to build a wall, 10 feet high, and 500 feet long, of a brick and half thick; reckoning the brick 10 inches long, and 4 courses to the foot in height? Ans. 72000.

QUEST. 19. How many bricks will build a square pyramid of 100 feet on each side at the base, and also 100 feet perpendicular height: the dimensions of a brick being supposed 10 inches long, 5 inches broad, and 3 inches thick?

Ans. 3840000.

QUEST. 20. If, from a right-angled triangle, whose base is 12, and perpendicular 16 feet, a line be drawn parallel to the

the perpendicular, cutting off a triangle whofe area is 24 fquare feet; required the fides of this triangle?

Anf. 6, 8, and 10.

QUEST. 21. The ellipfe in Grofvenor-fquare meafures 840 links acrofs the longeft way, and 612 the fhorteft, within the rails: now the walls being 14 inches thick, what ground do they inclofe, and what do they ftand upon?

Anf. { inclofe 4 ac. 0 r. 6 p. / ftand on $1760\frac{1}{2}$ fq. feet.

QUEST. 22. If a round pillar, 7 inches over, have 4 feet of ftone in it; of what diameter is the column, of equal length, that contains 10 times as much?

Anf. 22·136 inches.

QUEST. 23. A circular fifh-pond is to be made in a garden, that fhall take up juft half an acre; what muft be the length of the chord that ftrikes the circle? Anf. $27\frac{3}{4}$ yards.

QUEST. 24. When a roof is of a true pitch, or making a right angle at the ridge, the rafters are nearly $\frac{3}{4}$ of the breadth of the building: now fuppofing the eaves-boards to project 10 inches on a fide, what will the new ripping a houfe coft, that meafures 32 feet 9 inches long, by 22 feet 9 inches broad on the flat, at 15s. per fquare? Anf. 8l. 15s. $9\frac{1}{2}$d.

QUEST. 25. A cable, which is 3 feet long, and 9 inches in compafs, weighs 22lb; what will a fathom of that cable weigh, which meafures a foot about? Anf. $78\frac{2}{3}$lb.

QUEST. 26. My plumber has put 28lb. per fquare foot into a ciftern, 74 inches and twice the thicknefs of the lead long, 26 inches broad, and 40 deep; he has alfo put three ftays acrofs it within, of the fame ftrength, and 16 inches deep, and reckons 22s. per cwt, for work and materials. I, being a mafon, have paved him a workfhop, 22 feet 10 inches broad, with Purbeck ftone, at 7d. per foot; and on the balance I find there is 3s. 6d. due to him; what was the length of the workfhop, fuppofing fheet lead of $\frac{1}{10}$ of an inch thick to weigh 5·899lb. the fquare foot. Anf. 32 feet, $0\frac{1}{4}$ inch.

QUEST. 27. The diftance of the centres of two circles, whofe diameters are each 50, being given, equal to 30; what is the area of the fpace inclofed by their circumferences? Anf. 559·119.

QUEST. 28. If 20 feet of iron railing weigh half a ton, when the bars are an inch and quarter fquare; what will 50 feet come to at $3\frac{1}{2}$d. per lb, the bars being but $\frac{7}{8}$ of an inch fquare? Anf. 20l. 0s. 2d.

QUEST. 29.

QUEST. 29. The area of an equilateral triangle, whofe bafe falls on the diameter, and its vertex in the middle of the arc of a femicircle, is equal to 100: what is the diameter of the femicircle? Anf. 26·32148.

QUEST. 30. It is required to find the thicknefs of the lead in a pipe, of an inch and quarter bore, which weighs 14lb per yard in length; the cubic foot of lead weighing 11325 ounces? Anf. ·20737 inches.

QUEST. 31. Suppofing the expence of paving a femicircular plot, at 2s. 4d. per foot, come to 10l; what is the diameter of it? Anf. 14·7737 feet.

QUEST. 32. What is the length of a chord which cuts off $\frac{1}{3}$ of the area from a circle whofe diameter is 289? Anf. 278·6716.

QUEST. 33. My plumber has fet me up a ciftern, and, his fhop-book being burnt, he has no means of bringing in the charge, and I do not choofe to take it down to have it weighed; but by meafure he finds it contains $64\frac{3}{10}$ fquare feet, and that it is precifely $\frac{1}{8}$ of an inch in thicknefs. Lead was then wrought at 21l. per fother of $19\frac{1}{2}$ cwt. It is required from thefe items to make out the bill, allowing $6\frac{4}{9}$ oz. for the weight of a cubic inch of lead? Anf. 4l. 11s. 2d.

QUEST. 34. What will the diameter of a globe be, when the folidity and fuperficial content are expreffed by the fame number? Anf. 6.

QUEST. 35. A fack, that would hold 3 bufhels of corn, is $22\frac{1}{2}$ inches broad when empty; what will another fack contain, which, being of the fame length, has twice its breadth or circumference? Anf. 12 bufhels.

QUEST. 36. A carpenter is to put an oaken curb to a round well, at 8d. per foot fquare; the breadth of the curb is to be $7\frac{1}{4}$ inches, and the diameter within $3\frac{1}{2}$ feet; what will be the expence? Anf. 5s. $2\frac{1}{4}$d.

QUEST. 37. A gentleman has a garden 100 feet long, and 80 feet broad; and a gravel walk is to be made of an equal width half round it: what muft the breadth of the walk be, to take up juft half the ground? Anf. 25·968 feet.

QUEST. 38. The top of a may-pole, being broken off by a blaft of wind, ftruck the ground at 15 feet diftance from the foot of the pole; what was the height of the whole may-pole, fuppofing the length of the broken piece to be 39 feet? Anf. 75 feet.

QUEST. 39.

QUEST. 39. Seven men bought a grinding ſtone, of 60 inches diameter, each paying $\frac{1}{7}$ part of the expence; what part of the diameter muſt each grind down for his ſhare?

Anſ. the 1ſt 4·4508, 2d 4·8400, 3d 5·3535, 4th 6·0765, 5th 7·2079, 6th 9·3935, 7th 22·6778 inches.

QUEST. 40. A maltſter has a kiln, that is 16 feet 6 inches ſquare: but he wants to pull it down, and build a new one, that may dry three times as much at once as the old one; what muſt be the length of its ſide? Anſ. 28 feet, 7 inches.

QUEST. 41. How many 3 inch cubes may be cut out of a 12 inch cube? Anſ. 64.

QUEST. 42. How long muſt the tether of a horſe be, that will allow him to graze, quite around, juſt an acre of ground? Anſ. 39$\frac{1}{4}$ yards.

QUEST. 43. What will the painting of a conical ſpire come to, at 8d. per yard; ſuppoſing the height to be 118 feet, and the circumference of the baſe 64 feet? Anſ. 14l. 0s. 8$\frac{3}{4}$d.

QUEST. 44. The diameter of a ſtandard corn buſhel is 18$\frac{1}{2}$ inches, and its depth 8 inches; then what muſt the diameter of that buſhel be, whoſe depth is 7$\frac{1}{2}$ inches?

Anſ. 19·1067 inches.

QUEST. 45. Suppoſe the ball on the top of St. Paul's church is 6 feet in diameter; what did the gilding of it coſt at 3$\frac{1}{2}$d per ſquare inch? Anſ. 237l. 10s. 1d.

QUEST. 46. What will a fruſtum of a marble cone come to, at 12s. per ſolid foot; the diameter of the greater end being 4 feet, that of the leſs end 1$\frac{1}{2}$, and the length of the ſlant ſide 8 feet? Anſ. 30l. 1s. 10$\frac{1}{4}$d.

QUEST. 47. To divide a cone into three equal parts by ſections parallel to the baſe, and to find the altitudes of the three parts, the height of the whole cone being 20 inches?

Anſ. the upper part 13·867
the middle part 3·605
the lower part 2·528

QUEST. 48. A gentleman has a bowling-green, 300 feet long, and 200 feet broad, which he would raiſe 1 foot higher, by means of the earth to be dug out of a ditch that goes round it: to what depth muſt the ditch be dug, ſuppoſing its breadth to be every where 8 feet? Anſ. 7$\frac{2}{8}\frac{3}{6}$ feet.

QUEST. 49. How high above the earth muſt a perſon be raiſed, that he may ſee $\frac{1}{3}$ of its ſurface?

Anſ. to the height of the earth's diameter.

QUEST. 50.

QUEST. 50. A cubic foot of braſs is to be drawn into wire, of $\frac{1}{20}$ of an inch in diameter; what will the length of the wire be, allowing no loſs in the metal?

Anſ. 97784·797 yards, or 55 miles 984·797 yards.

QUEST. 51. Of what diameter muſt the bore of a cannon be, which is caſt for a ball of 24lb weight, ſo that the diameter of the bore may be $\frac{1}{10}$ of an inch more than that of the ball? Anſ. 5·647 inches.

QUEST. 52. Suppoſing the diameter of an iron 9lb ball to be 4 inches, as it is very nearly; it is required to find the diameters of the ſeveral balls weighing 1, 2, 3, 4, 6, 12, 18, 24, 32, 36, and 42lb, and the caliber of their guns, allowing $\frac{1}{50}$ of the caliber, or $\frac{1}{49}$ of the ball's diameter, for windage. Anſwer.

Wt ball.	Diameter ball.	Caliber gun.
1	1·9230	1·9622
2	2·4228	2·4723
3	2·7734	2·8301
4	3·0526	3·1149
6	3·4943	3·5656
9	4·0000	4·0816
12	4·4026	4·4924
18	5·0397	5·1425
24	5·5469	5·6601
32	6·1051	6·2297
36	6·3496	6·4792
42	6·6844	6·8208

QUEST. 53. Suppoſing the windage of all mortars to be $\frac{1}{50}$ of the caliber, and the diameter of the hollow part of the ſhell to be $\frac{7}{10}$ of the caliber of the mortar: it is required to determine the diameter and weight of the ſhell, and the quantity or weight of powder requiſite to fill it, for each of the ſeveral ſorts of mortars, namely, the 13, 10, 8, 5·8, and 4·6 inch mortar. Anſwer.

Calib. mort.	Diameter ſhell.	Wt ſhell empty.	Wt of powder.	Wt ſhell filled.
4·6	4·523	8·320	0·583	8·903
5·8	5·703	16·677	1·168	17·845
8	7·867	43·764	3·065	46·829
10	9·833	85·476	5·986	91·462
13	12·783	187·791	13·151	200·942

QUES. 54.

Quest. 54. If a heavy fphere, whofe diameter is 4 inches, be let fall into a conical glafs, full of water, whofe diameter is 5, and altitude 6 inches; it is required to determine how much water will run over?

Anf. 26·272 cubic inches, or nearly $\frac{3}{4}$ of a pint.

Quest. 55. The dimenfions of the fphere and cone being the fame as in the laft queftion, and the cone only $\frac{1}{5}$ full of water; required what part of the axis of the fphere is immerfed in the water? Anf. ·546 parts of an inch.

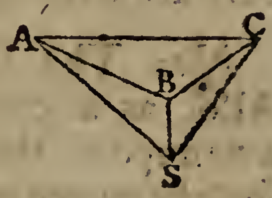
Quest. 56. The cone being ftill the fame, and $\frac{2}{5}$ full of water; required the diameter of a fphere which fhall be juft all covered by the water? Anf. 2·445996 inches.

Quest. 57. If a perfon, with an air balloon, afcend vertically from London, to fuch height that he can juft fee Oxford appear in the horizon; it is required to determine his height above the earth, fuppofing its circumference to be 25000 miles, and the diftance between London and Oxford 49·5933 miles? Anf. $\frac{311}{1000}$ of a mile, or 547 yards 1 foot.

Quest. 58. In a garrifon there are three remarkable objects, A, B, C, the diftances of which from one to another are known to be, AB 213, AC 424, and BC 262 yards; I am defirous of knowing my pofition and diftance at a place or ftation S, from whence I obferved the angle ASB 13° 30′, and the angle CSB 29° 50′, both by geometry and trigonometry.

Anfwer.

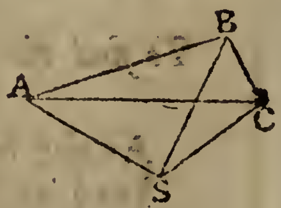
AS 605·7122;
BS 429·6814;
CS 524·2365.



Quest. 59. Required the fame as in the laft queftion, when the point B is on the other fide of AC, fuppofing AB 9, AC 12, and BC 6 furlongs; alfo the angle ASB 33° 45′, and the angle BSC 22° 30′.

Anfwer.

AS 10·64, BS 15·64, CS 14·01.



Quest. 60. It is required to determine the magnitude of a cube of gold, of the ftandard finenefs, which fhall be equal

equal to a ſum of 480 million of pounds ſterling; ſuppoſing a guinea to weigh 5 dwts $9\frac{1}{2}$ grains. Anſ. 18·691 feet.

QUEST. 61. The ditch of a fortification is 1000 feet long, 9 feet deep, 20 feet broad at bottom, and 22 at top; how much water will fill the ditch?

Anſ. 1158127 gallons nearly.

QUEST. 62. If the diameter of the earth be 7930 miles, and that of the moon 2160 miles: required the ratio of their ſurfaces, and alſo of their ſolidities; ſuppoſing them both to be globular, as they are very nearly?

Anſ. the ſurfaces are as $13\frac{1}{2}$ to 1 nearly; and the ſolidities as $49\frac{1}{2}$ to 1 nearly.

PRACTICAL EXERCISES CONCERNING SPECIFIC GRAVITY.

THE Specific Gravities of Bodies, are their relative weights contained under the ſame given magnitude; as a cubic foot, or a cubic inch, &c.

The ſpecific gravities of ſeveral ſorts of matter, are expreſſed by the numbers annexed to their names in the Table of Specific Gravities, at page 221; from whence the numbers are to be taken, when wanted.

Note. The ſeveral ſorts of wood are ſuppoſed to be dry. Alſo, as a cubic foot of water weighs juſt 1000 ounces averdupois, the numbers in the table expreſs, not only the ſpecific gravities of the ſeveral bodies, but alſo the weight of a cubic foot of each in averdupois ounces; and hence, by proportion, the weight of any other quantity, or the quantity of any other weight, may be known, as in the following problems:

PROBLEM I.

To find the Magnitude of any Body, from its Weight.

As the tabular ſpecific gravity of the body,
Is to its weight in averdupois ounces,
So is one cubic foot, or 1728 cubic inches,
To its content in feet, or inches, reſpectively.

EXAM-

EXAMPLES.

EXAM. 1. Required the content of an irregular block of common ſtone, which weighs 1cwt, or 112lb.

Anſ. $1228\frac{4}{5}$ cubic inches.

EXAM. 2. How many cubic inches of gunpowder are there in 1lb weight? Anſ. 30 cubic inches nearly.

EXAM. 3. How many cubic feet are there in a ton weight of dry oak? Anſ. $38\frac{138}{185}$ cubic feet.

PROBLEM II.

To find the Weight of a Body, from its Magnitude.

As one cubic foot, or 1728 cubic inches,
Is to the content of the body,
So is its tabular ſpecific gravity,
To the weight of the body.

EXAMPLES.

EXAM. 1. Required the weight of a block of marble, whoſe length is 63 feet, and breadth and thickneſs each 12 feet; being the dimenſions of one of the ſtones in the walls of Balbeck?

Anſ. $683\frac{7}{16}$ ton, which is nearly equal to the burthen of an Eaſt-India ſhip.

EXAM. 2. What is the weight of 1 pint, ale meaſure, of gunpowder? Anſ. 19 oz. nearly.

EXAM. 3. What is the weight of a block of dry oak, which meaſures 10 feet in length, 3 feet broad, and $2\frac{1}{2}$ feet deep? Anſ. $4335\frac{5}{16}$lb.

PROBLEM III.

To find the Specific Gravity of a Body.

CASE 1. When the body is heavier than water, weigh it both in water and out of water, and take the difference, which will be the weight loſt in water. Then ſay,

As the weight loſt in water,
Is to the whole weight,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE.

A piece of ſtone weighed 10lb, but in water only $6\frac{1}{6}$lb, required its ſpecific gravity? Anſ. 2609.

CASE

Case 2. When the body is lighter than water, ſo that it will not quite ſink; affix to it a piece of another body, heavier than water, ſo that the maſs compounded of the two may ſink together. Weigh the denſer body and the compound maſs, ſeparately, both in water, and out of it; then find how much each loſes in water, by ſubtracting its weight in water from its weight in air; and ſubtract the leſs of theſe remainders from the greater. Then ſay,

As the laſt remainder,
Is to the weight of the light body in air,
So is the ſpecific gravity of water,
To the ſpecific gravity of the body.

EXAMPLE.

Suppoſe a piece of elm weighs 15lb in air; and that a piece of copper, which weighs 18lb in air, and 16lb in water, is affixed to it, and that the compound weighs 6lb in water; required the ſpecific gravity of the elm? Anſ. 600.

PROBLEM IV.

To find the Quantities of Two Ingredients in a Given Compound.

Take the three differences of every pair of the three ſpecific gravities, namely, the ſpecific gravities of the compound and each ingredient; and multiply the difference of every two ſpecific gravities by the third. Then, as the greateſt product, is to the whole weight of the compound, ſo is each of the other products, to the two weights of the ingredients.

EXAMPLE.

A compoſition of 112lb being made of tin and copper, whoſe ſpecific gravity is found to be 8784; required the quantity of each ingredient, the ſpecific gravity of tin being 7320, and of copper 9000?

Anſwer, there is 100lb of copper and conſequently 12lb of tin } in the compoſition.

Of

OF THE WEIGHT AND DIMENSIONS OF BALLS AND SHELLS.

THE weight and dimenſions of Balls and Shells might be found from the problems laſt given, concerning ſpecific gravity. But they may be found ſtill eaſier by means of the experimented weight of a ball of a given ſize, from the known proportion of ſimilar figures, namely, as the cubes of their diameters.

PROBLEM I.

To find the Weight of an Iron Ball, from its Diameter.

An iron ball of 4 inches diameter weighs 9lb, and the weights being as the cubes of the diameters, it will be, as 64 (which is the cube of 4) is to 9, ſo is the cube of the diameter of any other ball, to its weight. Or, take $\frac{9}{64}$ of the cube of the diameter, for the weight. Or, take $\frac{1}{8}$ of the cube of the diameter, and $\frac{1}{8}$ of that again, and add the two together, for the weight.

EXAMPLES.

EXAM. 1. The diameter of an iron ſhot being 6·7 inches, required its weight? Anſ. 42·294lb.

EXAM. 2. What is the weight of an iron ball, whoſe diameter is 5·54 inches? Anſ. 24lb.

PROBLEM II.

To find the Weight of a Leaden Ball.

A leaden ball of $4\frac{1}{4}$ inches diameter weighs 17lb; therefore, as the cube of $4\frac{1}{4}$ is to 17, or nearly as 9 is to 2, ſo is the cube of the diameter of a leaden ball, to its weight.

Or, take $\frac{2}{9}$ of the cube of the diameter, for the weight, nearly.

EXAMPLES.

EXAM. 1. Required the weight of a leaden ball of 6·6 inches diameter? Anſ. 63·888lb.

EXAM. 2. What is the weight of a leaden ball of 5·24 inches diameter? Anſ. 32lb nearly.

PRO-

PROBLEM III.

To find the Diameter of an Iron Ball.

Multiply the weight by $7\frac{1}{5}$, and the cube root of the product will be the diameter.

EXAMPLES.

Exam. 1. Required the diameter of a 42lb iron ball? Anf. 6·685 inches.

Exam. 2. What is the diameter of a 24lb iron ball? Anf. 5·54 inches.

PROBLEM IV.

To find the Diameter of a Leaden Ball.

Multiply the weight by 9, and divide the product by 2; then the cube root of the quotient will be the diameter.

EXAMPLES.

Exam. 1. Required the diameter of a 64lb leaden ball? Anf. 6·604 inches.

Exam. 2. What is the diameter of an 8lb leaden ball? Anf. 3·302 inches.

PROBLEM V.

To find the Weight of an Iron Shell.

Take $\frac{9}{64}$ of the difference of the cubes of the external and internal diameter, for the weight of the fhell.

That is, from the cube of the external diameter, take the cube of the internal diameter, multiply the remainder by 9, and divide the product by 64.

EXAMPLES.

Exam. 1. The outfide diameter of an iron fhell being 12·8, and the infide diameter 9·1 inches; required its weight? Anf. 188·941lb.

Exam. 2. What is the weight of an iron fhell, whofe external and internal diameters are 9·8 and 7 inches? Anf. $84\frac{1}{8}$lb.

PRO-

PROBLEM VI.

To find how much Powder will fill a Shell.

Divide the cube of the internal diameter, in inches, by 57·3, for the lbs of powder.

EXAMPLES.

EXAM. 1. How much powder will fill the ſhell whoſe internal diameter is 9·1 inches? Anſ. $13\frac{2}{13}$ lb nearly.

EXAM. 2. How much powder will fill the ſhell whoſe internal diameter is 7 inches? Anſ. 6lb.

PROBLEM VII.

To find how much Powder will fill a Rectangular Box.

FIND the content of the box in inches, by multiplying the length, breadth, and depth all together. Then divide by 30 for the pounds of powder.

EXAMPLES.

EXAM. 1. Required the quantity of powder that will fill a box, the length being 15 inches, the breadth 12, and the depth 10 inches? Anſ. 60lb.

EXAM. 2. How much powder will fill a cubical box whoſe ſide is 12 inches? Anſ. $57\frac{3}{5}$ lb.

PROBLEM VIII.

To find how much Powder will fill a Cylinder.

MULTIPLY the ſquare of the diameter by the length, then divide by 38·2 for the pounds of powder.

EXAMPLES.

EXAM. 1. How much powder will the cylinder hold, whoſe diameter is 10 inches, and length 20 inches? Anſ. $52\frac{1}{3}$ lb nearly.

EXAM. 2. How much powder can be contained in the cylinder, whoſe diameter is 4 inches, and length 12 inches? Anſ. $5\frac{5}{191}$ lb.

PROBLEM IX.

To find the Size of a Shell to contain a given Weight of Powder.

MULTIPLY the pounds of powder by 57·3, and the cube root of the product will be the diameter in inches.

EXAMPLES.

EXAM. 1. What is the diameter of a ſhell that will hold $13\frac{1}{6}$ lb of powder? Anſ. 9·1 inches.

EXAM. 2. What is the diameter of a ſhell to contain 6lb of powder? Anſ. 7 inches.

PROBLEM X.

To find the Size of a Cubical Box, to contain a given Weight of Powder.

MULTIPLY the weight in pounds by 30, and the cube root of the product will be the ſide of the box in inches.

EXAMPLES.

EXAM. 1. Required the ſide of a cubical box, to hold 50lb of gunpowder? Anſ. 11·44 inches.

EXAM. 2. Required the ſide of a cubical box, to hold 400lb of gunpowder? Anſ. 22·89 inches.

PROBLEM XI.

To find what Length of a Cylinder will be filled by a given Weight of Gunpowder.

MULTIPLY the weight in pounds by 38·2, and divide the product by the ſquare of the diameter in inches, for the length.

EXAMPLES.

EXAM. 1. What length of a 36 pounder gun, of $6\frac{2}{3}$ inches diameter, will be filled with 12lb of gunpowder? Anſ. 10·314 inches.

EXAM. 2. What length of a cylinder, of 8 inches diameter, may be filled with 20lb of powder? Anſ. $11\frac{15}{16}$ inches.

OF

OF THE PILING OF BALLS AND SHELLS.

IRON Balls and Shells are commonly piled, by horizontal courſes, either in a pyramidical or wedge-like form ; the baſe being either an equilateral triangle, a ſquare, or a rectangle. In the triangle and ſquare, the pile will finiſh in a ſingle ball; but in the rectangle, it will finiſh in a ſingle row of balls, like an edge.

In triangular and ſquare piles, the number of horizontal rows, or courſes, is always equal to the number of balls in one ſide of the bottom row. And in rectangular piles, the number of rows is equal to the number of balls in the breadth of the bottom row. Alſo, the number in the top row, or edge, is one more than the difference between the length and breadth of the bottom row.

PROBLEM I.

To find the Number of Balls in a Triangular Pile.

MULTIPLY continually together, the number in one ſide of the bottom row, that number increaſed by 1, and the ſame number increaſed by 2 ; and $\frac{1}{6}$ of the laſt product will be the anſwer.

That is, $\frac{n . \overline{n+1} . \overline{n+2}}{6}$ is the number or ſum ; where n is the number in the bottom row.

EXAMPLES.

EXAM. 1. Required the number of balls in a triangular pile, each ſide of the baſe containing 30 balls? Anſ. 4960.

EXAM. 2. How many balls are in the triangular pile, each ſide of the baſe containing 20? Anſ. 1540.

PROBLEM II.

To find the Number of Balls in a Square Pile.

MULTIPLY continually together, the number in one ſide of the bottom courſe, that number increaſed by 1, and double the ſame number increaſed by 1 ; then $\frac{1}{6}$ of the laſt product will be the anſwer.

That is, $\frac{n . \overline{n+1} . \overline{2n+1}}{6}$ is the number.

EXAMPLES.

EXAM. 1. How many balls are in a ſquare pile of 30 rows? Anſ. 9455.

EXAM. 2. How many balls are in a ſquare pile of 20 rows? Anſ. 2870.

PROBLEM III.

To find the Number of Balls in a Rectangular Pile.

FROM 3 times the number in the length of the baſe row, ſubtract one leſs than the breadth of the ſame; multiply the remainder by the ſaid breadth, and the product by one more than the ſame; and divide by 6 for the anſwer.

That is, $\frac{b \,.\, \overline{b+1} \,.\, \overline{3l-b+1}}{6}$ is the number; where l is the length, and b the breadth, of the loweſt courſe.

Note. In all the piles, the breadth of the bottom is equal to the number of courſes. And, in the oblong or rectangular pile, the top row is one more than the difference between the length and breadth of the bottom.

EXAMPLES.

EXAM. 1. Required the number of balls in a rectangular pile, the length and breadth of the baſe row being 46 and 15? Anſ. 4960.

EXAM. 2. How many ſhot are in a rectangular complete pile, the length of the bottom courſe being 59, and its breadth 20? Anſ. 11060.

PROBLEM IV.

To find the Number of Balls in an Incomplete Pile.

FROM the number in the whole pile, conſidered as complete, ſubtract the number in the upper pile which is wanting at the top, both computed by the rule for their proper form; and the remainder will be the number in the fruſtum, or incomplete pile.

EXAMPLES.

EXAM. 1. To find the number of ſhot in the incomplete triangular pile, one ſide of the bottom courſe being 40, and the top courſe 20? Anſ. 10150.

EXAM. 2. How many ſhot are in the incomplete triangular pile, the ſide of the baſe being 24, and of the top 8? Anſ. 2516.

EXAM. 3. How many balls are in the incomplete ſquare pile, the ſide of the baſe being 24, and of the top 8? Anſ. 4760.

EXAM. 4. How many ſhot are in the incomplete rectangular pile, of 12 courſes, the length and breadth of the baſe being 40 and 20? Anſ. 6146.

OF

OF DISTANCES BY THE VELOCITY OF SOUND.

BY various experiments it has been found, that ſound flies, through the air, uniformly at the rate of about 1142 feet in 1 ſecond of time, or a mile in $4\frac{2}{3}$ ſeconds. And therefore, by proportion, any diſtance may be found correſponding to any given time; namely, multiplying the given time, in ſeconds, by 1142, for the correſponding diſtance in feet; or taking $\frac{3}{14}$ of the given time for the diſtance in miles.

Note. The time for the paſſage of ſound in the interval between ſeeing the flaſh of a gun, or lightning, and hearing the report, may be obſerved by a watch, or a ſmall pendulum. Or, it may be obſerved by the beats of the pulſe in the wriſt, counting, on an average, about 70 to a minute for perſons in moderate health, or $5\frac{1}{2}$ pulſations to a mile; and more or leſs according to circumſtances.

EXAMPLES.

EXAM. 1. After obſerving a flaſh of lightning, it was 12 ſeconds before the thunder was heard; required the diſtance of the cloud from whence it came? Anſ. $2\frac{4}{7}$ miles.

EXAM. 2. How long, after firing the Tower guns, may the report be heard at Shooter's-Hill, ſuppoſing the diſtance to be 8 miles in a ſtraight line? Anſ. $37\frac{1}{3}$ ſeconds.

EXAM. 3. After obſerving the firing of a large cannon at a diſtance, it was 7 ſeconds before the report was heard; what was its diſtance? Anſ. $1\frac{1}{2}$ mile.

EXAM. 4. Perceiving a man at a diſtance hewing down a tree with an axe, I remarked that 6 of my pulſations paſſed between ſeeing him ſtrike and hearing the report of the blow; what was the diſtance between us, allowing 70 pulſes to a minute? Anſ. 1 mile and 198 yards.

EXAM. 5. How far off was the cloud from which thunder iſſued, whoſe report was 5 pulſations after the flaſh of lightning; counting 75 to a minute? Anſ. 1523 yards.

EXAM. 6. If I ſee the flaſh of a cannon, fired by a ſhip in diſtreſs at ſea, and hear the report 33 ſeconds after, how far is ſhe off? Anſ. $7\frac{1}{14}$ miles.

PRAC-

PRACTICAL EXERCISES IN MECHANICS, STATICS, HYDROSTATICS, SOUND, MOTION, GRAVITY, PROJECTILES, and other Branches of Natural Philosophy.

QUESTION 1. REQUIRED the weight of a cast iron ball of 3 inches diameter, supposing the weight of a cubic inch of the metal to be 0·2581b averdupois? Ans. 3·64739lb.

QUEST. 2. To determine the weight of a hollow spherical iron shell, 5 inches in diameter, the thickness of the metal being one inch? Ans. 13·2387 lb.

QUEST. 3. Being one day ordered to observe how far a battery of cannon was from me, I counted, by my watch, 17 seconds between the time of seeing the flash and hearing the report; what then was the distance? Ans. $3\frac{2}{3}$ miles.

QUEST. 4. It is proposed to determine the proportional quantities of matter in the earth and moon; the density of the former being to that of the latter, as 10 to 7, and their diameters as 7930 to 2160. Ans. as 71 to 1 nearly.

QUEST. 5. What difference is there, in point of weight, between a block of marble containing 1 cubic foot and a half, and another of brass of the same dimensions?

Ans. 496lb 14oz.

QUEST. 6. In the walls of Balbeck in Turkey, the ancient Heliopolis, there are three stones laid end to end, now in sight, that measure in length 61 yards; one of which in particular is 21 yards or 63 feet long, 12 feet thick, and 12 feet broad: now if this block be marble, what power would balance it, so as to prepare it for moving?

Ans. $683\frac{7}{16}$ tons, the burthen of an East-India ship.

QUEST. 7. The battering-ram of Vespasian weighed, suppose 100000 pounds; and was moved, let us admit, with such a velocity, by strength of hand, as to pass through 20 feet in one second of time; and this was found sufficient to demolish the walls of Jerusalem. The question is, with what velocity a 32lb ball must move, to do the same execution? Ans. 62500 feet.

QUEST. 8. There are two bodies, of which the one contains 25 times the matter of the other, or is 25 times heavier; but the less moves with 1000 times the velocity of the

the greater: in what proportion then are the momenta, or forces, with which they are moved?

Anſ. the leſs moves with a force 40 times greater.

QUEST. 9. A body, weighing 20lb, is impelled by ſuch a force, as to ſend it through 100 feet in a ſecond; with what velocity then would a body of 8lb weight move, if it were impelled by the ſame force? Anſ. 250 feet per ſecond.

QUEST. 10. There are two bodies, the one of which weighs 100lb, the other 60; but the leſs body is impelled by a force 8 times greater than the other; the proportion of the velocities, with which theſe bodies move, is required? Anſ. the velocity of the greater to that of the leſs, as 3 to 40.

QUEST. 11. There are two bodies, the greater contains 8 times the quantity of matter in the leſs, and is moved with a force 48 times greater; the ratio of the velocities of theſe two bodies is required?

Anſ. the greater is to the leſs, as 6 to 1.

QUEST. 12. There are two bodies, one of which moves 40 times ſwifter than the other; but the ſwifter body has moved only one minute, whereas the other has been in motion 2 hours: the ratio of the ſpaces deſcribed by theſe two bodies is required?

Anſ. the ſwifter is to the ſlower, as 1 to 3.

QUEST. 13. Suppoſing one body to move 30 times ſwifter than another, as alſo the ſwifter to move 12 minutes, the other only 1: what difference will there be between the ſpaces deſcribed by them, ſuppoſing the laſt has moved 60 inches? Anſ. 1795 feet.

QUEST. 14. There are two bodies, the one of which has paſſed over 50 miles, the other only 5; and the firſt had moved with 5 times the celerity of the ſecond: what is the ratio of the times they have been in deſcribing thoſe ſpaces?

Anſ. as 2 to 1.

QUEST. 15. If a lever, 40 effective inches long, will, by a certain power thrown ſucceſſively on it, in 13 hours, raiſe a weight 104 feet; in what time will two other levers, each 18 effective inches long, raiſe an equal weight 73 feet?

Anſ. 10 hours $8\frac{1}{3}$ minutes.

QUEST. 16. What weight will a man be able to raiſe, who preſſes with the force of a hundred and a half, on the end of an equipoiſed handſpike, 100 inches long, meeting with a con-

a convenient prop exactly $7\frac{1}{2}$ inches from the lower end of the machine? Anſ. 2072lb.

Quest. 17. A weight of $1\frac{1}{2}$lb, laid on the ſhoulder of a man, is no greater burthen to him than its abſolute weight, or 24 ounces: what difference will he feel, between the ſaid weight applied near his elbow, at 12 inches from the ſhoulder, and in the palm of his hand, 28 inches from the ſame; and how much more muſt his muſcles then draw, to ſupport it at right angles, that is, having his arm ſtretched right out?
Anſ. 24lb averdupois.

Quest. 18. What weight hung on at 70 inches from the centre of motion of a ſteel-yard, will balance a ſmall gun of $9\frac{1}{2}$ cwt, freely ſuſpended at 2 inches diſtance from the ſaid centre on the contrary ſide? Anſ. $30\frac{2}{5}$lb.

Quest. 19. It is propoſed to divide the beam of a ſteel-yard, or to find the points of diviſion where the weights of 1, 2, 3, 4, &c, lb, on the one ſide, will juſt balance a conſtant weight of 95lb at the diſtance of 2 inches on the other ſide of the fulcrum; the weight of the beam being 10lb, and its whole length 36 inches?
Anſ. 30, 15, 10, $7\frac{1}{2}$, 6, 5, $4\frac{2}{7}$, $3\frac{3}{4}$, $3\frac{1}{3}$, 3, 2 8, $2\frac{1}{2}$, &c.

Quest. 20. Two men carrying a burthen of 200lb weight between them, hung on a pole, the ends of which reſt on their ſhoulders; how much of this load is borne by each man, the weight hanging 6 inches from the middle, and the whole length of the pole being 4 feet? Anſ. 125lb and 75lb.

Quest. 21. If, in a pair of ſcales, a body weigh 90lb in one ſcale, and only 40lb in the other; required its true weight, and the proportion of the lengths of the two arms of the balance beam, on each ſide of the point of ſuſpenſion?
Anſ. the weight 60lb, and the proportion 3 to 2.

Quest. 22. To find the weight of a beam of timber, or other body, by means of a man's own weight, or any other weight. For inſtance, a piece of tapering timber, 24 feet long, being laid over a prop, or the edge of another beam, is found to balance itſelf when the prop is 13 feet from the leſs end; but removing the prop a foot nearer to the ſaid end, it takes a man's weight of 210lb, ſtanding on the leſs end, to hold it in equilibrium. Required the weight of the tree? Anſ. 2520lb.

Quest. 23. If AB be a cane or walking-ſtick, 40 inches long, ſuſpended by a ſtring SD faſtened to the middle point D: now

D: now a body being hung on at E, 6 inches diſtance from D, is balanced by a weight of 2lb, hung on at the larger end A; but removing the body to F, one inch nearer to D, the 2lb weight on the other ſide is moved to G, within 8 inches of D, before the cane will reſt in equilibrio. Required the weight of the body? Anſ. 24lb.

QUEST. 24. If AB, BC be two inclined planes, of the lengths of 30 and 40 inches, and moveable about the joint at B: what will be the ratio of two weights P, Q, in equilibrio on the planes, in all poſitions of them: and what will be the altitude BD of the angle B above the horizontal plane AC, when this is 50 inches long?

Anſ. BD = 24; and P to Q as AB to BC, or as 3 to 4.

QUEST. 25. A lever, of 6 feet long, is fixed at right angles in a ſcrew, whoſe threads are one inch aſunder, ſo that the lever turns juſt once round in raiſing or depreſſing the ſcrew one inch. If then this lever be urged by a weight or force of 50lb, with what force will the ſcrew preſs?

Anſ. $22619\frac{1}{2}$lb.

QUEST. 26. If a man can draw a weight of 150lb up the ſide of a perpendicular wall, of 20 feet high; what weight will he be able to raiſe along a ſmooth plank of 30 feet long, laid aſlope from the top of the wall? Anſ. 225lb.

QUEST. 27. If a force of 150lb be applied on the head of a rectangular wedge, its thickneſs being 2 inches, and the length of its ſide 12 inches; what weight will it raiſe or balance perpendicular to its ſide? Anſ. 900lb.

QUEST. 28. If a round pillar, of 30 feet diameter, be raiſed on a plane, inclined to the horizon in an angle of 75°, or the ſhaft inclining 15 degrees out of the perdendicular; what length will it bear before it overſet?

Anſ. $30(2 + \sqrt{3})$ or 111·9615 feet.

QUEST. 29. If the greateſt angle at which a bank of natural earth will ſtand, be 45°; it is propoſed to determine what thickneſs an upright wall of ſtone muſt be made throughout, juſt to ſupport a bank of 12 feet high; the ſpecific gravity of the ſtone being to that of earth, as 5 to 4.

Anſ. $12\sqrt{\frac{8}{15}}$, or 8·76356 feet.

QUEST. 30. If the ſtone wall be made like a wedge or having its upright ſection a triangle, tapering to a point at top, but its ſide next the bank of earth perpendicular to the horizon; what is its thickneſs at the bottom, ſo as to ſupport the ſame bank? Anſ. $12\sqrt{\frac{4}{5}}$, or 10·733126 feet.

QUEST. 31.

QUEST. 31. But if the earth will only ſtand at an angle of 30 degrees to the horizontal line; it is required to determine the thickneſs of wall in both the preceding caſes?

Anſ. the breadths are the ſame as before, becauſe the area of the triangular bank of earth is increaſed in the ſame proportion as its horizontal puſh is decreaſed.

QUEST. 32. To find the thickneſs of an upright rectangular wall, neceſſary to ſupport a body of water; the water being 10 feet deep, and the wall 12 feet high; alſo the ſpecific gravity of the wall to that of the water, as 11 to 7.

Anſ. 4·204374 feet.

QUEST. 33. To determine the thickneſs of the wall at the bottom, when the ſection of it is triangular, and the altitudes as before. Anſ. 5·1492866 feet.

QUEST. 34. Suppoſing the diſtance of the earth from the ſun to be 95 millions of miles; I would know at what diſtance from him another body muſt be placed, ſo as to receive light and heat quadruple to that of the earth.

Anſ. at half the diſtance, or $47\frac{1}{2}$ millions.

QUEST. 35. If the mean diſtance of the ſun from us be 106 of his diameters; how much hotter is it at the ſurface of the ſun, than under our equator?

Anſ. 11236 times hotter.

QUEST. 36. The diſtance between the earth and ſun being accounted 95 millions of miles, and between Jupiter and the ſun 495 millions; the degree of light and heat received by Jupiter, compared with that of the earth, is required?

Anſ. $\frac{361}{9801}$, or nearly $\frac{1}{27}$ of the earth's light and heat.

QUEST. 37. A certain body on the ſurface of the earth weighs a cwt, or 112lb; the queſtion is, whither this body muſt be carried, that it may weigh only 10lb?

Anſ. either at 3·3466 ſemi-diameters, or $\frac{5}{56}$ of a ſemi-diameter, from the centre.

QUEST. 38. If a body weigh 1 pound, or 16 ounces, on the ſurface of the earth; what will its weight be at 50 miles above it, taking the earth's diameter at 7930 miles?

Anſ. 15 oz. $9\frac{5}{8}$ dr. nearly.

QUEST. 39. Whereabouts, in the line between the earth and moon, is their common centre of gravity; ſuppoſing the earth's diameter to be 7930 miles, and the moon's 2160; alſo the denſity of the former to that of the latter, as 99 to 68, or

68, or as 10 to 7 nearly, and their mean diſtance 30 of the earth's diameters?

Anſ. at $\frac{105}{251}$ parts of a diameter from the earth's centre, or $\frac{41}{502}$ parts of a diameter, or 648 miles below the ſurface.

QUEST. 40. Whereabouts, between the earth and moon, are their attractions equal to each other? Or, where muſt another body be placed, ſo as to remain ſuſpended in equilibrio, not being more attracted to the one than to the other, or having no tendency to fall either way? Their dimenſions being as in the laſt queſtion.

Anſ. $\left.\begin{array}{ll}\text{From the earth's centre} & 26\frac{9}{11} \\ \text{From the moon's centre} & 3\frac{2}{11}\end{array}\right\}$ of the earth's diameters.

QUEST. 41. Suppoſe a ſtone, dropt into an abyſs, ſhould be ſtopped at the end of the 11th ſecond after its delivery; what ſpace would it have gone through? Anſ. $1946\frac{1}{12}$ feet.

QUEST. 42. What is the difference between the depths of two wells, into each of which ſhould a ſtone be dropped at the ſame inſtant, the one will ſtrike the bottom at 6 ſeconds, the other at 10? Anſ. $1029\frac{1}{3}$ feet.

QUEST. 43. If a ſtone be $19\frac{1}{2}$ ſeconds in deſcending from the top of a precipice to the bottom, what is its height?
Anſ. $6115\frac{11}{16}$ feet.

QUEST. 44. In what time will a muſket ball, dropped from the top of Saliſbury ſteeple, ſaid to be 400 feet high, reach the bottom? Anſ. 5 ſeconds nearly.

QUEST. 45. If a heavy body be obſerved to fall through 100 feet in the laſt ſecond of time, from what height did it fall, and how long was it in motion?
Anſ. time $3\frac{235}{386}$ ſec. and height $209\frac{4273}{9264}$ feet.

QUEST. 46. A ſtone being let fall into a well, it was obſerved that, after being dropped, it was 10 ſeconds before the ſound of the fall at the bottom reached the ear. What is the depth of the well? Anſ. 1270 feet nearly.

QUEST. 47. It is propoſed to determine the length of a pendulum vibrating ſeconds, in the latitude of London, where a heavy body falls through $16\frac{1}{12}$ feet in the firſt ſecond of time? Anſ. 39·11 inches.

By experiment this length is found to be $39\frac{1}{8}$ inches.

QUEST. 48. What is the length of a pendulum vibrating in 2 ſeconds; alſo in half a ſecond, and in a quarter ſecond?

Anſ. the 2 ſecond pendulum $156\frac{1}{2}$
the $\frac{1}{2}$ ſecond pendulum $9\frac{25}{32}$
the $\frac{1}{4}$ ſecond pendulum $2\frac{57}{128}$ inches.

QUEST. 49.

QUEST. 49. What difference will there be in the number of vibrations, made by a pendulum of 6 inches long, and another of 12 inches long, in an hour's time? Anf. $2692\frac{1}{2}$.

QUEST. 50. Obferved that while a ftone was defcending, to meafure the depth of a well, a ftring and plummet, that from the point of fufpenfion, or the place where it was held, to the centre of ofcillation, meafured juft 18 inches, had made 8 vibrations, when the found from the bottom returned. What was the depth of the well? Anf. 412·61 feet.

QUEST. 51. If a ball vibrate in the arch of a circle, 10 degrees on each fide of the perpendicular; or a ball roll down the loweft 10 degrees of the arch; required the velocity at the loweft point? the radius of the circle, or length of the pendulum, being 20 feet. Anf. 4·4213 feet per fecond.

QUEST. 52. If a ball defcend down a fmooth inclined plane, whofe length is 100 feet, and altitude 10 feet; how long will it be in defcending, and what will be the laft velocity?

Anf. the veloc. 25·364 feet per fec. and time 7·8852 fec.

QUEST. 53. If a cannon ball, of 1lb weight, be fired againft a pendulous block of wood, and, ftriking the centre of ofcillation, caufe it to vibrate an arc whofe chord is 30 inches; the radius of that arc, or diftance from the axis to the loweft point of the pendulum, being 118 inches, and the pendulum vibrating in fmall arcs 40 ofcillations per minute. Required the velocity of the ball, and the velocity of the centre of ofcillation of the pendulum, at the loweft point of the arc; the whole weight of the pendulum being 500lb?

Anf. veloc. ball 1956·6054 feet per fec.
and veloc. cent. ofcil. 3·9054 feet per fec.

QUEST. 54. How deep will a cube of oak fink in common water; each fide of the cube being 1 foot?

Anf. $11\frac{1}{10}$ inches.

QUEST. 55. How deep will a globe of oak fink in water; the diameter being 1 foot? Anf. 9·9867 inches.

QUEST. 56. If a cube of wood, floating in common water, have 3 inches of its height dry above the water, and $4\frac{8}{103}$ inches dry when in fea-water; it is propofed to determine the magnitude of the cube, and what fort of wood it is made of? Anf. the wood is oak, and each fide 40 inches.

QUEST. 57. An irregular piece of lead ore weighs, in air 12 ounces, but in water only 7; and another fragment weighs

weighs in air $14\frac{1}{2}$ ounces, but in water only 9; required their comparative denſities, or ſpecific gravities?

Anſ. as 145 to 132.

Quest. 58. An irregular fragment of glaſs, in the ſcale, weighs 171 grains, and another of magnet 102 grains; but in water the firſt fetches up no more than 120 grains, and the other 79: what then will their ſpecific gravities turn out to be? Anſ. glaſs to magnet as 3933 to 5202 or nearly as 10 to 13.

Quest. 59. Hiero, king of Sicily, ordered his jeweller to make him a crown, containing 63 ounces of gold. The workmen thought that, ſubſtituting part ſilver was only a proper perquiſite; which taking air, Archimedes was appointed to examine it; who, on putting it into a veſſel of water, found it raiſed the fluid 8·2245 cubic inches: and having diſcovered that the inch of gold more critically weighed 10·36 ounces, and that of ſilver but 5·85 ounces, he found by calculation what part of the king's gold had been changed. And you are deſired to repeat the proceſs. Anſ. 28·8 ounces.

Quest. 60. Suppoſing the cubic inch of common glaſs weigh 1·4921 ounces troy, the ſame of ſea-water ·59542, and of brandy ·5368; then a ſeaman having a gallon of this liquor in a glaſs bottle, which weighs 3·84lb out of water, and, to conceal it from the officers of the cuſtoms, throws it overboard. It is propoſed to determine, if it will ſink, how much force will juſt buoy it up? Anſ. 14·1496 ounces.

Quest. 61. Another perſon has half an anker of brandy, of the ſame ſpecific gravity as in the laſt queſtion; the wood of the caſk ſuppoſe meaſures $\frac{1}{8}$ of a cubic foot; it is propoſed to aſſign what quantity of lead is juſt requiſite to keep the caſk and liquor under water? Anſ. 89·743 ounces.

Quest. 62. Suppoſe, by meaſurement, it be found that a man of war, with its ordnance, rigging, and appointments, ſinks ſo deep as to diſplace 50000 cubic feet of freſh water; what is the whole weight of the veſſel? Anſ. $1395\frac{1}{10}$ tons.

Quest. 63. It is required to determine what would be the height of the atmoſphere, if it were every where of the ſame denſity as at the ſurface of the earth, when the quickſilver in the barometer ſtands at 30 inches; and alſo, what would be the height of a water barometer at the ſame time?

Anſ. height of the air $29166\frac{2}{3}$ feet or 5·5240 miles, height of water 35 feet.

Quest. 64.

QUEST. 64. With what velocity would each of thofe three fluids, viz. quickfilver, water, and air, iffue through a fmall orifice in the bottom of veffels, of the refpective heights of 30 inches, 35 feet, and 5·5240 miles; eftimating the preffure by the half altitudes, and the air rufhing into a vacuum?

Anf. the veloc. of quickfilver		8·967 feet.
the veloc. of water	-	33·55
the veloc. of air	- -	968·6

But, eftimating by the whole altitude, the veloc. of air is - - - - - - 1369·8
And the mean between thefe two is - 1169·2
which is nearly the velocity of found, and alfo nearly equal to the velocity of a ball through the air when it fuffers a refiftance equal to the preffure of the atmofphere.

QUEST. 65. A very large veffel of 10 feet high (no matter what fhape) being kept conftantly full of water, by a large fupplying cock at the top; if 9 fmall circular holes, each $\frac{1}{5}$ of an inch diameter, be opened in its perpendicular fide at every foot of the depth: it is required to determine the feveral diftances to which they will fpout on the horizontal plane of the bafe, and the quantity of water difcharged by all of them in 10 minutes?

Anf. the diftances are

$\sqrt{36}$	or	6·00000
$\sqrt{64}$	-	8·00000
$\sqrt{84}$	-	9·16515
$\sqrt{96}$	-	9·79796
$\sqrt{100}$	-	10·00000
$\sqrt{96}$	-	9·79796
$\sqrt{84}$	-	9·16515
$\sqrt{64}$	-	8·00000
$\sqrt{36}$	-	6·00000

and the quantity difcharged in 10 min. 123·8849 gallons.

Note. In this folution, the velocity of the water is fuppofed to be equal to that which is acquired by a heavy body in falling through the whole height of the water above the orifice, and that it is the fame in every part of the holes.

QUEST. 66. If the inner axis of a hollow globe of copper, exhaufted of air, be 100 feet; what thicknefs muft it be of, that it may juft float in air?

Anf. ·02688 of an inch thick.

QUEST. 67. If a ſpherical balloon of copper, of $\frac{1}{100}$ of an inch thick, have its cavity of 100 feet diameter, and be filled with inflammable air, of $\frac{1}{10}$ of the gravity of common air, what weight will juſt balance it, and prevent it from riſing up into the atmoſphere? Anſ. 27981lb.

QUEST. 68. If a glaſs tube, 36 inches long, cloſe at top, be ſunk perpendicularly into water, till its lower or open end be 30 inches below the ſurface of the water; how high will the water riſe within the tube, the quickſilver in the common barometer at the ſame time ſtanding at $29\frac{1}{2}$ inches?

Anſ. 2·26545 inches.

QUEST. 69. If a diving bell, of the form of a parabolic conoid, be let down into the ſea to the ſeveral depths of 5, 10, 15, and 20 fathoms; it is required to aſſign the reſpective heights to which the water will riſe within it: its axis and the diameter of its baſe being each 8 feet, and the quickſilver in the barometer ſtanding at 30·9 inches?

Anſ. at 5 fathoms deep the water riſes 2·03546 feet,
at 10 - - - - 3·06393
at 15 - - - - 3·70267
at 20 - - - - 4.14658.

THE

The DOCTRINE of FLUXIONS.

DEFINITIONS AND PRINCIPLES.

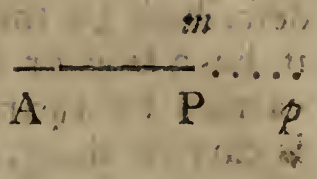
Art. 1. In the Doctrine of Fluxions, magnitudes or quantities of all kinds, are conſidered, not as made up of a number of ſmall parts, but as generated by continued motion, by means of which they increaſe or decreaſe. As, a line by the motion of a point; a ſurface by the motion of a line; and a ſolid by the motion of a ſurface. So likewiſe, time may be conſidered as repreſented by a line, increaſing uniformly by the motion of a point. And quantities of all kinds whatever, which are capable of increaſe and decreaſe, may, in like manner, be repreſented by lines, ſurfaces, or ſolids, conceived to be generated by motion.

2. Any quantity, thus generated, and variable, is called a Fluent, or a Flowing Quantity. And the rate or proportion according to which any flowing quantity increaſes, at any poſition or inſtant, is the Fluxion of the ſaid quantity, at that poſition or inſtant: and it is proportional to the magnitude by which the flowing quantity would be uniformly increaſed, in a given time with the generating celerity uniformly continued during that time.

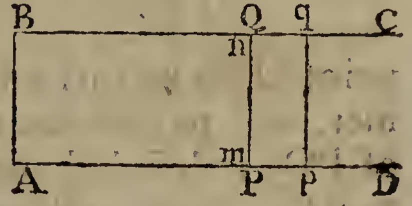
3. The ſmall quantities that are actually generated, produced, or deſcribed, in any ſmall given time, and by any continued motion, either uniform or variable, are called Increments.

4. Hence, if the motion of increaſe be uniform, by which increments are generated, the increments will in that caſe be proportional, or equal, to the meaſures of the fluxions: but if the motion of increaſe be accelerated, the increment ſo generated, in a given finite time, will exceed the fluxion: and if it be a decreaſing motion, the increment, ſo generated, will be leſs than the fluxion. But if the time be indefinitely ſmall, ſo that the motion be conſidered as uniform for that inſtant; then theſe naſcent increments will always be proportional, or equal, to the fluxions, and may be ſubſtituted inſtead of them, in any calculation.

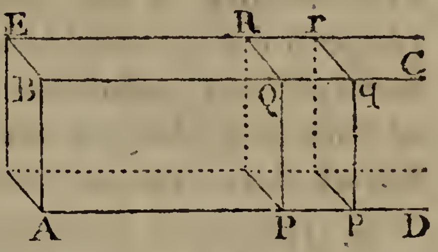
5. To



5. To illustrate these definitions: Suppose a point m be conceived to move from the position A, and to generate a line AP, by a motion any how regulated; and suppose the celerity of the point m, at any position P, to be such, as would, if from thence it should become or continue uniform, be sufficient to cause the point to describe, or pass uniformly over, the distance Pp, in the given time allowed for the fluxion: then will the said line Pp represent the fluxion of the fluent, or flowing line, AP, at that position.



6. Again, suppose the right line mn to move, from the position AB, continually parallel to itself, with any continued motion, so as to generate the fluent or flowing rectangle ABQP, whilst the point m describes the line AP: also, let the distance Pp be taken, as before, to express the fluxion of the line or base AP; and complete the rectangle PQqp. Then, like as Pp is the fluxion of the line AP, so is Pq the fluxion of the flowing parallelogram AQ; both these fluxions, or increments, being uniformly described in the same time.

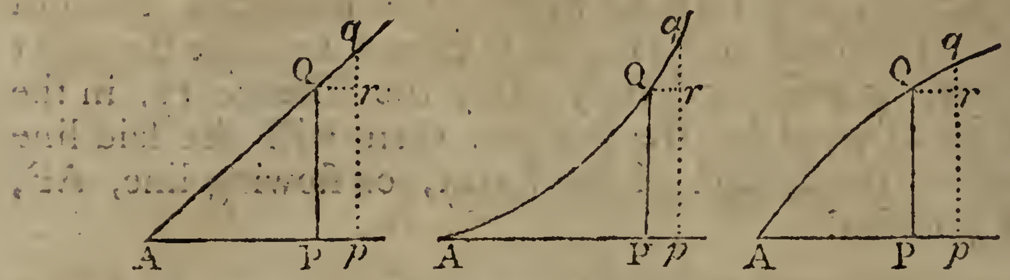


7. In like manner, if the solid AERP be conceived to be generated by the plane PQR moving, from the position ABE, always parallel to itself, along the line AD; and if Pp denote the fluxion of the line AP: Then, like as the rectangle PQqp, or PQ $\times$ Pp, expresses the fluxion of the flowing rectangle ABQP, so also shall the fluxion of the variable solid, or prism ABERQP, be expressed by the prism PQRrqp, or the plane PR $\times$ Pp. And, in both these last two cases, it appears, that the fluxion of the generated rectangle, or prism, is equal to the product of the generating line, or plane, drawn into the fluxion of the line along which it moves.

8. Hitherto the generating line, or plane, has been considered as of a constant and invariable magnitude; in which case the fluent, or quantity generated, is a rectangle, or a prism, the former being described by the motion of a line, and the latter by the motion of a plane. So, in like manner are other figures, whether plane or solid, conceived to be

 described

described, by the motion of a Variable Magnitude, whether it be a line or a plane. Thus let a variable line PQ be carried by a parallel motion along AP, or whilst a point P is carried along, and describes the line AP, suppose another



point Q to be carried by a motion perpendicular to the former, and to describe the line PQ: let *pq* be another position of PQ, indefinitely near to the former; and draw Q*r* parallel to AP. Now, in this case, there are several fluents, or flowing quantities, with their respective fluxions: namely, the line or fluent AP, the fluxion of which is P*p* or Q*r*; the line or fluent PQ, the fluxion of which is *rq*; the curve or oblique line AQ, described by the oblique motion of the point Q, the fluxion of which is Q *q*; and lastly, the surface APQ, described by the variable line PQ, the fluxion of which is the rectangle PQ*rp*, or PQ × P*p*. In the same manner may any solid be conceived to be described, by the motion of a variable plane parallel to itself, substituting the variable plane for the variable line; in which case, the fluxion of the solid, at any position, is represented by the variable plane, at that position, drawn into the fluxion of the line along which it is carried.

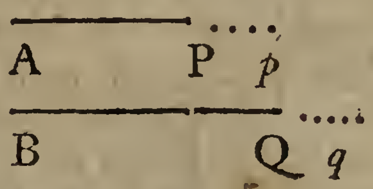
9. Hence then it follows in general, that the fluxion of any figure, whether plane or solid, at any position, is equal to the section of it, at that position, drawn into the fluxion of the axis, or line along which the variable section is supposed to be perpendicularly carried; that is, the fluxion of the figure AQP, is equal to the plane PQ × P*p*, when that figure is a solid, or to the ordinate PQ × P*p*, when the figure is a surface.

10. It also follows, from the same premises, that, in any curve, or oblique line, AQ, whose abscis is AP, and ordinate is PQ, the fluxions of these three form a small right-angled plane triangle Q *qr*; for Q*r* = P*p* is the fluxion of the abscis AP, *qr* the fluxion of the ordinate PQ, and Q *q* the fluxion of the curve or right line AQ. And consequently that, in any curve, the square of the fluxion of the curve, is

equal

equal to the ſum of the ſquares of the fluxions of the abſciſs and ordinate, when theſe two are at right angles to each other.

11. From the premiſes it alſo appears, that contemporaneous fluents, or quantities that flow or increaſe together; which are always in a conſtant ratio to each other, have their fluxions alſo in the ſame conſtant ratio, at every poſition. For, let AP and BQ be two contemporaneous fluents, deſcribed in the ſame time by the motion of the points P and Q, the contemporaneous poſitions being P, Q, and p, q; and let AP be to BQ, or Ap to Bq, conſtantly in the ratio of 1 to n.



Then is $n \times AP = BQ$,
and $n \times Ap = Bq$;
therefore, by ſubtraction, $n \times Pp = Qq$;
that is, the fluxion Pp : fluxion Qq :: 1 : n,
the ſame as the fluent AP : fluent BQ :: 1 : n;
or, the fluxions and fluents are in the ſame conſtant ratio.

But if the ratio of the fluents be variable, ſo will that of the fluxions be alſo, though not in the ſame variable ratio with the former, at every poſition.

NOTATION, &c.

12. To apply the foregoing principles to the determination of the fluxions of algebraical quantities, by means of which thoſe of all other kinds are determined, it will be neceſſary firſt to premiſe the notation commonly uſed in this ſcience, with ſome obſervations. As, firſt, that the final letters of the alphabet z, y, x, w, &c, are uſed to denote variable or flowing quantities; and the initial letters, a, b, c, d, &c, to denote conſtant or invariable ones: Thus, the variable baſe AP of the flowing rectangular figure ABQP, in art. 6, may be repreſented by x; and the invariable altitude PQ, by a: alſo, the variable baſe or abſciſs AP, of the figures in art. 8, may be repreſented by x; the variable ordinate PQ, by y; and the variable curve or line AQ, by z.

Secondly, that the fluxion of a quantity denoted by a ſingle letter, is repreſented by the ſame letter with a point over it: Thus, the fluxion of x is expreſſed by $\dot{x}$, the fluxion of y by $\dot{y}$, and the fluxion of z by $\dot{z}$. As to the fluxions of conſtant or invariable quantities, as of a, b, c, &c, they are equal to nothing, becauſe they do not flow or change their magnitude.

Thirdly, that the increments of variable or flowing quantities, are alſo denoted by the ſame letters with a ſmall ′ over them: Thus, the increments of x, y, z, are x', y', z'.

13. From theſe notations, and the foregoing principles, the quantities, and their fluxions, there conſidered, will be denoted as below. Thus, in all the foregoing figures put

the variable or flowing line - - AP $= x$,
in art. 6, the conſtant line - - PQ $= a$,
in art. 8, the variable ordinate - PQ $= y$,
alſo, the variable line or curve - AQ $= z$:

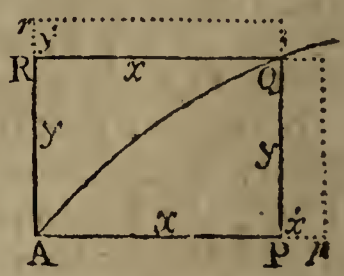
Then ſhall the ſeveral fluxions be thus repreſented, namely,

$\dot{x} =$ Pp the fluxion of the line AP,
$a\dot{x} =$ PQqp the fluxion of ABQP in art. 6,
$y\dot{x} =$ PQrp the fluxion of APQ in art. 8,
$\dot{z} =$ Qq $= \sqrt{\dot{x}^2 + \dot{y}^2}$ the fluxion of AQ; and
$a\dot{x} =$ Pr the fluxion of the ſolid in art. 7, if a denote the conſtant generating plane PQR. Alſo,
$nx =$ BQ in the figure to art. 11, and
$n\dot{x} =$ Qq the fluxion of the ſame.

14. The principles and notation being now laid down, we may proceed to the practice and rules of this doctrine, which conſiſts of two principal parts, called the Direct and Inverſe Method of Fluxions; namely, the direct method, which conſiſts in finding the fluxion of any propoſed fluent or flowing quantity; and the inverſe method, which conſiſts in finding the fluent of any propoſed fluxion. As to the former of theſe two problems, it can always be determined, and that in finite algebraical terms; but the latter, or finding of fluents, can only be effected in ſome certain caſes, except by means of infinite ſeries.—Firſt then, of

The DIRECT METHOD of FLUXIONS.

To find the Fluxion of the Product or Rectangle of two Variable Quantities.



15. Let ARQP, $= xy$, be the flowing or variable rectangle, generated by two lines PQ and RQ, moving always perpendicular to each other, from the poſitions AR and AP; denoting the one by x, and the other by y; ſuppoſing x and y to be ſo related, that the curve line AQ may always paſs through the interſection Q of thoſe lines, or the oppoſite angle of the rectangle.

Now,

Now, the rectangle consists of the two trilinear spaces APQ, ARQ, of which the

fluxion of the former is PQ × P$\dot{p}$, or $y\dot{x}$, and

that of the latter is - RQ × Rr, or $x\dot{y}$, by art. 8;

therefore the sum of the two, $\dot{x}y + x\dot{y}$, is the fluxion of the whole rectangle xy or ARQP.

The Same Otherwise.

16. Let the sides of the rectangle, x and y, by flowing, become $x + \acute{x}$ and $y + \acute{y}$: then the product of these two, or $xy + x\acute{y} + y\acute{x} + \acute{x}\acute{y}$ will be the new or contemporaneous value of the flowing rectangle PR or xy; subtract the one value from the other, and the remainder, $x\acute{y} + y\acute{x} + \acute{x}\acute{y}$, will be the increment generated in the same time as $\acute{x}$ or $\acute{y}$; of which the last term $\acute{x}\acute{y}$ is nothing, or indefinitely small, in respect of the other two terms, because $\acute{x}$ and $\acute{y}$ are indefinitely small in respect of x and y; which term being therefore omitted, there remains $x\acute{y} + y\acute{x}$ for the value of the increment: and hence, by substituting $\dot{x}$ and $\dot{y}$ for $\acute{x}$ and $\acute{y}$, to which they are proportional, there arises $x\dot{y} + y\dot{x}$ for the true value of the fluxion of xy; the same as before.

17. Hence may be easily derived the fluxion of the powers and products of any number of flowing or variable quantities whatever; as of xyz, or $wxyz$, or $vwxyz$, &c. And, first, for the fluuion of xyz: put $p = xy$, and the whole given fluent $xyz = q$, or $q = xyz = pz$. Then, taking the fluxions of $q = pz$, by the last article, they are $\dot{q} = \dot{p}z + p\dot{z}$; but $p = xy$, and so $\dot{p} = \dot{x}y + x\dot{y}$ by the same article; substituting therefore these values of p and $\dot{p}$ instead of them, in the value of $\dot{q}$, this becomes $\dot{q} = \dot{x}yz + x\dot{y}z + xy\dot{z}$, the fluxion of xyz required; which is therefore equal to the sum of the products, arising from the fluxion of each letter, or quantity, multiplied by the product of the other two.

Again, to determine the fluxion of $wxyz$, the continual product of four variable quantities; put this product, namely $wxyz$, or qw, $= r$, where $q = xyz$ as above. Then, taking the fluxions by the last article, $\dot{r} = \dot{q}w + q\dot{w}$; which, by substituting for q and $\dot{q}$ their values as above, becomes - - -

$\dot{r} = \dot{w}xyz + w\dot{x}yz + wx\dot{y}z + wxy\dot{z}$, the fluxion of $wxyz$ as required: consisting of the fluxion of each quantity, drawn into the products of the other three.

In

In the very same manner it is found, that the fluxion of $vwxyz$ is $\dot{v}wxyz + v\dot{w}xyz + vw\dot{x}yz + vwx\dot{y}z + vwxy\dot{z}$; and so on, for any number of quantities whatever; in which it is always found, that there are as many terms as there are variable quantities in the proposed fluent; and that these terms consist of the fluxion of each variable quantity, multiplied by the product of all the rest of the quantities.

18. From hence is easily derived the fluxion of any power of a variable quantity, as of x^2, or x^3, or x^4, &c. For, in the product or rectangle xy, if $x = y$, then is $xy = xx$ or x^2, and also its fluxion $\dot{x}y + x\dot{y} = \dot{x}x + x\dot{x}$ or $2x\dot{x}$, the fluxion of x^2.

Again, if all the three x, y, z be equal; then is the product of the three $xyz = x^3$; and consequently its fluxion $\dot{x}yz + x\dot{y}z + xy\dot{z} = \dot{x}xx + x\dot{x}x + xx\dot{x}$ or $3x^2\dot{x}$, the fluxion of x^3.

In the same manner, it will appear that
the fluxion of x^4 is $= 4x^3\dot{x}$,
the fluxion of x^5 is $= 5x^4\dot{x}$, and, in general,
the fluxion of x^n is $= nx^{n-1}\dot{x}$;
where n is any positive whole number whatever.

That is, the fluxion of any positive integral power, is equal to the fluxion of the root ($\dot{x}$), multiplied by the exponent of the power (n), and by the power of the same root whose index is less by 1, (x^{n-1}).

And thus, the fluxion of $a + cx$ being $c\dot{x}$,
that of $(a + cx)^2$ is $2c\dot{x} \times (a + cx)$ or $2ac\dot{x} + 2c^2x\dot{x}$,
that of $(a + cx^2)^2$ is $4cx\dot{x} \times (a + cx^2)$ or $4acx\dot{x} + 4c^2x^3\dot{x}$,
that of $(x^2 + y^2)^2$ is $(4x\dot{x} + 4y\dot{y}) \times (x^2 + y^2)$,
that of $(x + cy^2)^3$ is $(3\dot{x} + 6cy\dot{y}) \times (x + cy^2)^2$.

19. From the conclusions in the same article, we may also derive the fluxion of any fraction, or the quotient of one variable quantity divided by another, as of $\frac{x}{y}$. For, put the quotient or fraction $\frac{x}{y} = q$; then, multiplying by the denominator, $x = qy$; and, taking the fluxions,

$$\dot{x} = \dot{q}y + q\dot{y}, \text{ or } \dot{q}y = \dot{x} - q\dot{y}; \text{ and, by division,}$$

$$\dot{q} = \frac{\dot{x}}{y} - \frac{q\dot{y}}{y} = (\text{by substituting the value of } q, \text{ or } \frac{x}{y}),$$

$$\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2} = \frac{\dot{x}y - x\dot{y}}{y^2}, \text{ the fluxion of } \frac{x}{y}, \text{ as required.}$$

That

That is, the fluxion of any fraction, is equal to the fluxion of the numerator drawn into the denominator, minus the fluxion of the denominator drawn into the numerator, and the remainder divided by the square of the denominator. So that the fluxion of $\frac{ax}{y}$ is $a \times \frac{\dot{x}y - x\dot{y}}{y^2}$ or $\frac{a\dot{x}y - ax\dot{y}}{y^2}$.

20. Hence, too, is easily derived the fluxion of any negative integer power of a variable quantity, as of x^{-n}, or $\frac{1}{x^n}$, which is the same thing. For here the numerator of the fraction is 1, whose fluxion is nothing; and therefore, by the last article, the fluxion of such a fraction, or negative power, is barely equal to minus the fluxion of the denominator, divided by the square of the said denominator. That is, the fluxion of x^{-n}, or $\frac{1}{x^n}$, is $-\frac{nx^{n-1}\dot{x}}{x^{2n}}$ or $-\frac{n\dot{x}}{x^{n+1}}$ or $-nx^{-n-1}\dot{x}$; or the fluxion of any negative integer power of a variable quantity, as x^{-n}, is equal to the fluxion of the root, multiplied by the exponent of the power, and by the next power less by 1.

The same thing is otherwise obtained thus: Put the proposed fraction, or quotient $\frac{1}{x^n} = q$; then is $qx^n = 1$; and, taking the fluxions, we have

$\dot{q}x^n + qnx^{n-1}\dot{x} = 0$; hence $\dot{q}x^n = -qnx^{n-1}\dot{x}$; divide by x^n, then $\dot{q} = -\frac{qn\dot{x}}{x} =$ (by substituting $\frac{1}{x^n}$ for q), $\frac{-n\dot{x}}{x^{n+1}}$ or $=$ $-nx^{-n-1}\dot{x}$; the same as before.

Hence the fluxion of x^{-1} or $\frac{1}{x}$ is $-x^{-2}\dot{x}$, or $-\frac{\dot{x}}{x^2}$,

that of - x^{-2} or $\frac{1}{x^2}$ is $-2x^{-3}\dot{x}$ or $-\frac{2\dot{x}}{x^3}$,

that of - x^{-3} or $\frac{1}{x^3}$ is $-3x^{-4}\dot{x}$ or $-\frac{3\dot{x}}{x^4}$,

that of - ax^{-4} or $\frac{a}{x^4}$ is $-4ax^{-5}\dot{x}$ or $-\frac{4a\dot{x}}{x^5}$,

that of $(a+x)^{-1}$ or $\frac{1}{a+x}$ is $-(a+x)^{-2}\dot{x}$ or $-\frac{\dot{x}}{(a+x)^2}$,

and that of $c(a+3x^2)^{-2}$ or $\frac{c}{(a+3x^2)^2}$ is $-12cx\dot{x} \times (a+3x^2)^{-3}$,

or $-\frac{12cx\dot{x}}{(a+3x^2)^3}$.

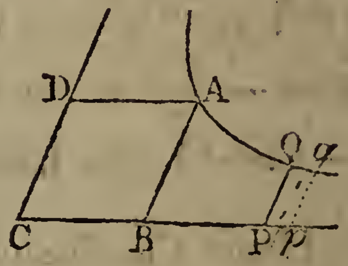
21. Much

21. Much in the ſame manner is obtained the fluxion of any fractional power of a fluent quantity, as of $x^{\frac{m}{n}}$, or $\sqrt[n]{x^m}$.

For, put the propoſed quantity $x^{\frac{m}{n}} = q$; then, raiſing each to the n power, gives $x^m = q^n$;
take the fluxions, ſo ſhall $mx^{m-1}\dot{x} = nq^{n-1}\dot{q}$; then
dividing by nq^{n-1}, gives $\dot{q} = \frac{mx^{m-1}\dot{x}}{nq^{n-1}} = \frac{mx^{m-1}\dot{x}}{nx^{m-\frac{m}{n}}} = \frac{m}{n}x^{\frac{m}{n}-1}\dot{x}$.
Which is ſtill the ſame rule, as before, for finding the fluxion of any power of a fluent quantity, and which therefore is general, whether the exponent be poſitive or negative, integral or fractional. And hence the fluxion of $ax^{\frac{3}{2}}$ is $\frac{3}{2}ax^{\frac{1}{2}}\dot{x}$, that of $ax^{\frac{1}{2}}$ is $\frac{1}{2}ax^{\frac{1}{2}-1}\dot{x} = \frac{1}{2}ax^{-\frac{1}{2}}\dot{x} = \frac{a\dot{x}}{2x^{\frac{1}{2}}} = \frac{a\dot{x}}{2\sqrt{x}}$, and that of

$\sqrt{a^2 - x^2}$ or $(a^2 - x^2)^{\frac{1}{2}}$ is $\frac{1}{2}(a^2 - x^2)^{-\frac{1}{2}} \times -2x\dot{x} = \frac{-x\dot{x}}{\sqrt{a^2 - x^2}}$.

22. Having now found out the fluxions of all the ordinary forms of algebraical quantities; it remains to determine thoſe of logarithmic expreſſions, and alſo of exponential ones, that is, ſuch powers as have their exponents variable or flowing quantities. And firſt, for the fluxion of Napier's, or the hyperbolic logarithm.



23. Now, to determine this from the nature of the hyperbolic ſpaces. Let A be the principal vertex of an hyperbola, having its aſymptotes CD, CP, with the ordinates DA, BA, PQ, &c, parallel to them. Then, from the nature of the hyperbola and of logarithms, it is known that any ſpace ABPQ is the log. of the ratio of CB to CP, to the modulus ABCD. Now, put $1 =$ CB or BA the ſide of the ſquare or rhombus DB; $m =$ the modulus, or CB $\times$ BA; or area of DB, or ſine of the angle C to the radius 1; alſo the abſciſs CP $= x$, and the ordinate PQ $= y$. Then, by the nature of the hyperbola, CP $\times$ PQ is always equal to DB, that is, $xy = m$; hence $y = \frac{m}{x}$, and the fluxion of the ſpace, or $\dot{x}y$ is $\frac{m\dot{x}}{x} =$ PQqp the fluxion of the log. of x, to the modulus m. And, in the hyperbolic logarithms, the modulus m being 1, therefore $\frac{\dot{x}}{x}$ is

$\frac{\dot{x}}{x}$ is the fluxion of the hyp. log. of x; which is therefore equal to the fluxion of the quantity, divided by the quantity itſelf.

Hence the fluxion of the hyp. log.

of $1 + x$ is $\frac{\dot{x}}{1+x}$,

of $1 - x$ is $\frac{-\dot{x}}{1-x}$,

of $x + z$ is $\frac{\dot{x}+\dot{z}}{x+z}$,

of $\frac{a+x}{a-x}$ is $\frac{\dot{x}(a-x)+\dot{x}(a+x)}{(a-x)^2} \times \frac{a-x}{a+x} = \frac{2a\dot{x}}{a^2-x^2}$,

of ax^n is $\frac{nax^{n-1}\dot{x}}{ax^n} = \frac{n\dot{x}}{x}$.

24. By means of the fluxions of logarithms, are uſually determined thoſe of exponential quantities, that is, quantities which have their exponent a flowing or variable letter. Theſe exponentials are of two kinds, namely, when the root is a conſtant quantity, as e^x, and when the root is variable as well as the exponent, as y^x

25. In the firſt caſe, put the exponential, whoſe fluxion is to be found, equal to a ſingle variable quantity z, namely, $z = e^x$; then take the logarithm of each, ſo ſhall log. $z =$ $x \times$ log. e; take the fluxions of theſe, ſo ſhall $\frac{\dot{z}}{z} = \dot{x} \times$ log. e by the laſt article; hence $\dot{z} = z\dot{x} \times$ log. $e = e^x\dot{x} \times$ log. e, which is the fluxion of the propoſed quantity e^x or z, and which therefore is equal to the ſaid given quantity drawn into the fluxion of the exponent, and into the log. of the root.

Hence alſo, the fluxion of $(a+c)^{nx}$ is $(a+c)^{nx} \times n\dot{x} \times$ log. $(a+c)$.

26. In like manner, in the ſecond caſe, put the given quantity $y^x = z$; then the logarithms give log. $z = x \times$ log. y, and the fluxions give $\frac{\dot{z}}{z} = \dot{x} \times$ log. $y + x \times \frac{\dot{y}}{y}$; hence $\dot{z} = z\dot{x} \times$ log. $y + \frac{zx\dot{y}}{y} =$ (by ſubſtituting y^x for z) $y^x\dot{x} \times$ log. $y + xy^{x-1}\dot{y}$, which is the fluxion of the propoſed quantity y^x; and which therefore conſiſts of two terms, of which the

the one is the fluxion of the given quantity confidering the exponent as conftant, and the other the fluxion of the fame quantity confidering the root as conftant.

Of SECOND, THIRD, &c, FLUXIONS.

HAVING explained the manner of confidering and determining the firft fluxions of flowing or variable quantities; it remains now to confider thofe of the higher orders, as fecond, third, fourth, &c, fluxions.

27. If the rate or celerity with which any flowing quantity changes its magnitude, be conftant, or the fame at every pofition; then is the fluxion of it alfo conftantly the fame. But if the variation of magnitude be continually changing, either increafing or decreafing; then will there be a certain degree of fluxion peculiar to every point or pofition; and the rate of variation or change in the fluxion, is called the Fluxion of the Fluxion, or the Second Fluxion of the given fluent quantity. In like manner, the variation or fluxion of this fecond fluxion, is called the Third Fluxion of the firft propofed fluent quantity; and fo on.

Thefe orders of fluxions are denoted by the fame fluent letter, with the correfponding number of points over it: namely, two points for the fecond fluxion, three points for the third fluxion, four points for the fourth fluxion, and fo on. So, the different orders of the fluxion of x, are $\dot{x}$, $\ddot{x}$, $\dddot{x}$, $\ddddot{x}$, &c; where each is the fluxion of the one next before it.

28. This defcription of the higher orders of fluxions may be illuftrated by the figures exhibited in page 274; where, if x denote the abfcifs AP, and y the ordinate PQ; and if the ordinate PQ or y flow along the abfcifs AP or x, with an uniform motion; then the fluxion of x, namely, $\dot{x} = Pp$ or Qr, is a conftant quantity, or $\ddot{x} = 0$, in all the figures. Alfo, in fig. 1, in which AQ is a right line, $\dot{y} = rq$, or the fluxion of PQ, is a conftant quantity, or $\ddot{y} = 0$; for, the angle Q, = the angle A, being conftant, Qr is to rq, or $\dot{x}$ to $\dot{y}$, in a conftant ratio. But in the 2d fig. rq, or the fluxion of PQ, continually increafes more and more; and in

in fig. 3 it continually decreaſes more and more, and therefore in both theſe caſes y has a ſecond fluxion, being poſitive in fig. 2, but negative in fig. 3. And ſo on, for the other orders of fluxions.

Thus, if, for inſtance, the nature of the curve be ſuch that x^3 is every where equal a^2y; then, taking the fluxions, it is $a^2\dot{y} = 3x^2\dot{x}$; and, conſidering $\dot{x}$ always as a conſtant quantity. and taking always the fluxions, the equations of the ſeveral orders of fluxions will be as below, viz.

the 1ſt fluxions $a^2\dot{y} = 3x^2\dot{x}$,

the 2d fluxions $a^2\ddot{y} = 6x\dot{x}^2$,

the 3d fluxions $a^2\dddot{y} = 6\dot{x}^3$,

the 4th fluxions $a^2\ddddot{y} = 0$,

and all the higher fluxions alſo $= 0$, or nothing.

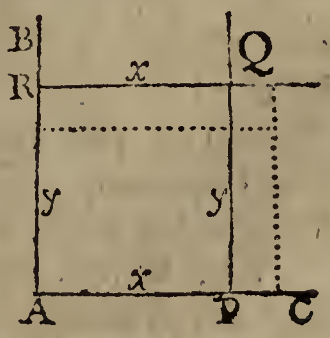
Alſo, the higher orders of fluxions are found in the ſame manner as the lower ones. Thus,

the firſt fluxion of y^3 is - - - - - $3y^2\dot{y}$;

its 2d flux. or the flux. of $3y^2\dot{y}$, conſidered as the rectangle of $3y^2$ and $\dot{y}$, is $\Big\}$ $3y^2\ddot{y} + 6y\dot{y}^2$;

and the flux. of this again, or the 3d flux. of y^3, is - - - - - $\Big\}$ $3\dddot{y}y^2 + 18y\dot{y}\ddot{y} + 6\dot{y}^3$.

29. In the foregoing articles, it has been ſuppoſed that the fluents increaſe, or that their fluxions are poſitive; but it often happens that ſome fluents decreaſe, and that therefore their fluxions are negative: and whenever this is the caſe, the ſign of the fluxion muſt be changed, or made contrary to that of the fluent. So, of the rectangle xy, when both x and y increaſe together, the fluxion is $\dot{x}y + x\dot{y}$; but if one of them, as y, decreaſe, while the other, x, increaſes; then, the fluxion of y being $-\dot{y}$, the fluxion of xy will in that caſe be $\dot{x}y - x\dot{y}$. This may be illuſtrated by the annexed rectangle APQR $= xy$, ſuppoſed to be generated by the motion of the line PQ from A towards C, and by the motion of the line RQ from B towards A: For, by the motion of PQ, from A towards C, the rectangle is increaſed, and its fluxion is $+\dot{x}y$; but, by the motion of RQ, from B towards A, the rectangle is decreaſed, and the fluxion of the decreaſe is $x\dot{y}$; therefore, taking



taking the fluxion of the decreafe from that of the increafe, the fluxion of the rectangle xy, when x increafes and y decreafes, is $\dot{x}y - x\dot{y}$.

30. We may now collect all the rules together, which have been demonftrated in the foregoing articles, for finding the fluxions of all forts of quantities. And,

1ft, *For the fluxion of Any Power of a flowing quantity.*—Multiply all together the exponent of the power, the fluxion of the root, and the power next lefs by 1 of the fame root.

2d, *For the fluxion of the Rectangle of two quantities.*—Multiply each quantity by the fluxion of the other, and connect the two products together by their proper figns.

3d, *For the fluxion of the Continual Product of any number of flowing quantities.*—Multiply the fluxion of each quantity by the product of all the other quantities, and connect all the products together by their proper figns.

4th, *For the fluxion of a Fraction.*—From the fluxion of the numerator drawn into the denominator, fubtract the fluxion of the denominator drawn into the numerator, and divide the refult by the fquare of the denominator.

5th, *Or, the 2d, 3d, and 4th cafes may be all included under one, and performed thus.*—Take the fluxion of the given expreffion as often as there are variable quantities in it, fuppofing firft only one of them variable, and the reft conftant; then another variable, and the reft conftant; and fo on, till they have all in their turns been fingly fuppofed variable: and connect all thefe fluxions together with their own figns.

6th, *For the fluxion of a Logarithm.*—Divide the fluxion of the quantity by the quantity itfelf, and multiply the refult by the modulus of the fyftem of logarithms.

Note, The modulus of the hyperbolic logarithms is 1, and the modulus of the common logs, is - 0·43429448.

7th, *For the fluxion of an Exponential quantity, having the root Conftant.*—Multiply all together, the given quantity, the fluxion of its exponent, and the hyp. log. of the root.

8th, *For the fluxion of an Exponential quantity having the root Variable.*—To the fluxion of the given quantity, found by the 1ft rule, as if the root only were variable, add the fluxion of the fame quantity, found by the 7th rule, as if the exponent only were variable; and the fum will be the fluxion for both of them variable.

Note, When the given quantity confifts of feveral terms, find the fluxion of each term feparately, and connect them all together with their proper figns.

31. PRAC-

31. PRACTICAL EXAMPLES TO EXERCISE THE FOREGOING RULES.

1. The fluxion of axy is
2. The fluxion of $bxyz$ is
3. The fluxion of $cx \times (ax - cy)$ is
4. The fluxion of $x^m y^n$ is
5. The fluxion of $x^m y^n z^r$ is
6. The fluxion of $(x + y) \times (x - y)$ is
7. The fluxion of $2ax^2$ is
8. The fluxion of $2x^3$ is
9. The fluxion of $3x^4 y$ is
10. The fluxion of $4x^{\frac{2}{3}} y^4$ is
11. The fluxion of $ax^2 y - x^{\frac{1}{2}} y^3$ is
12. The fluxion of $4x^4 - x^2 y + 3byz$ is
13. The fluxion of $\sqrt[n]{x}$ or $x^{\frac{1}{n}}$ is
14. The fluxion of $\sqrt[n]{x^m}$ or $x^{\frac{m}{n}}$ is
15. The fluxion of $\frac{1}{\sqrt[n]{x^m}}$ or $\frac{1}{x^{\frac{m}{n}}}$ or $x^{-\frac{m}{n}}$ is
16. The fluxion of $\sqrt{x}$ or $x^{\frac{1}{2}}$ is
17. The fluxion of $\sqrt[3]{\ }$ or $x^{\frac{1}{3}}$ is
18. The fluxion of $\sqrt[3]{x^2}$ or $x^{\frac{2}{3}}$ is
19. The fluxion of $\sqrt{x^3}$ or $x^{\frac{3}{2}}$ is
20. The fluxion of $\sqrt[4]{x^3}$ or $x^{\frac{3}{4}}$ is
21. The fluxion of $\sqrt[3]{x^4}$ or $x^{\frac{4}{3}}$ is
22. The fluxion of $\sqrt{a^2 + x^2}$ or $\overline{a^2 + x^2}^{\frac{1}{2}}$ is
23. The fluxion of $\sqrt{a^2 - x^2}$ or $\overline{a^2 - x^2}^{\frac{1}{2}}$ is
24. The fluxion of $\sqrt{2rx - xx}$ or $\overline{2rx - xx}^{\frac{1}{2}}$ is
25. The fluxion of $\frac{1}{\sqrt{a^2 - x^2}}$ or $(a^2 - x^2)^{-\frac{1}{2}}$ is
26. The fluxion of $\overline{ax - xx}^{\frac{1}{3}}$ is

27. The

27. The fluxion of $2x\sqrt{a^2 \pm x^2}$ is

28. The fluxion of $\overline{a^2 - x^2}^{\frac{3}{2}}$ is

29. The fluxion of $\sqrt{xz}$ or $xz^{\frac{1}{2}}$ is

30. The fluxion of $\sqrt{xz - zz}$ or $\overline{xz - zz}^{\frac{1}{2}}$ is

31. The fluxion of $-\frac{1}{a\sqrt{x}}$ or $-\frac{1}{a}x^{-\frac{1}{2}}$ is

32. The fluxion of $\frac{ax^3}{a + x}$ is

33. The fluxion of $\frac{x^m}{y^n}$ is

34. The fluxion of $\frac{xy}{z}$ is

35. The fluxion of $\frac{c}{xx}$ is

36. The fluxion of $\frac{3x}{a - x}$ is

37. The fluxion of $\frac{z}{x + z}$ is

38. The fluxion of $\frac{x^2}{z^2}$ is

39. The fluxion of $\frac{x^{\frac{2}{3}}}{y^{\frac{3}{2}}}$ is

40. The fluxion of $\frac{axy^2}{z}$ is

41. The fluxion of $\frac{3}{\sqrt{x^2 - y^2}}$ is

42. The fluxion of the hyp. log. of ax is

43. The fluxion of the hyp. log. of $1 + x$ is

44. The fluxion of the hyp. log. of $1 - x$ is

45. The fluxion of the hyp. log. of x^2 is

46. The fluxion of the hyp. log. of $\sqrt{z}$ is

47. The fluxion of the hyp. log. of x^m is

48. The

48. The fluxion of the hyp. log. of $\frac{2}{x^2}$ is

49. The fluxion of the hyp. log. of $\frac{1+x}{1-x}$ is

50. The fluxion of the hyp. log. of $\frac{1-x}{1+x}$ is

51. The fluxion of c^x is

52. The fluxion of 10^x is

53. The fluxion of $\overline{a+c}^x$ is

54. The fluxion of 100^{xy} is

55. The fluxion of x^z is

56. The fluxion of y^{10x} is

57. The fluxion of x^x is

58. The fluxion of $\overline{xy}^{xz}$ is

59. The fluxion of $x\dot{y}$ is

60. The fluxion of $\dot{x}\dot{y}^2$ is

61. The second fluxion of xy is

62. The second fluxion of xy, when $\dot{x}$ is constant, is

63. The second fluxion of x^n is

64. The third fluxion of x^n, when $\dot{x}$ is constant, is

65. The third fluxion of xy is

THE INVERSE METHOD, OR THE FINDING OF FLUENTS.

32. IT has been obferved, that a Fluent, or Flowing Quantity, is the variable quantity which is confidered as increafing or decreafing. Or, the fluent of a given fluxion, is fuch a quantity, that its fluxion, found according to the foregoing rules, fhall be the fame as the fluxion given or propofed.

33. It may farther be obferved, that Contemporary Fluents, or Contemporary Fluxions, are fuch as flow together, or for the fame time.—When contemporary fluents are always equal, or in any conftant ratio; then alfo are their fluxions refpectively either equal, or in that fame conftant ratio. That is, if $x = y$, then is $\dot{x} = \dot{y}$; or if $x : y :: n : 1$, then is $\dot{x} : \dot{y} :: n : 1$; or if $x = ny$, then is $\dot{x} = n\dot{y}$.

34. It

34. It is eafy to find the fluxions to all the given forms of fluents; but, on the contrary, it is difficult to find the fluents of many given fluxions; and indeed there are numberlefs cafes in which this cannot at all be done, excepting by the quadrature and rectification of curve lines, or by logarithms, or by infinite feries. For, it is only in certain particular forms and cafes that the fluents of given fluxions can be found; there being no method of performing this univerfally, *a priori*, by a direct inveftigation, like finding the fluxion of a given fluent quantity. We can only therefore lay down a few rules for fuch forms of fluxions as we know, from the direct method, belong to fuch and fuch kinds of flowing quantities: and thefe rules, it is evident, muft chiefly confift in performing fuch operations as are the reverfe of thofe by which the fluxions are found of given fluent quantities. The principal cafes of which are as follow.

35. *To find the Fluent of a Simple Fluxion; or of that in which there is no variable quantity, and only one fluxional quantity.*

This is done by barely fubftituting the variable or flowing quantity inftead of its fluxion; being the refult or reverfe of the notation only.—Thus,

The fluent of $a\dot{x}$ is ax.
The fluent of $a\dot{y} + 2\dot{y}$ is $ay + 2y$.
The fluent of $\overline{\sqrt{a^2 + x^2}}$ is $\sqrt{a^2 + x^2}$.

36. *When any Power of a flowing quantity is Multiplied by the Fluxion of the Root.*

Then, having fubftituted, as before, the flowing quantity for its fluxion, divide the refult by the new index of the power. Or, which is the fame thing, take out, or divide by, the fluxion of the root; add 1 to the index of the power; and divide by the index fo increafed. Which is the reverfe of the 1ft rule for finding fluxions.

So, if the fluxion propofed be - $3x^5\dot{x}$.
Leave out, or divide by, $\dot{x}$, then it is $3x^5$;
add 1 to the index, and it is - $3x^6$;
divide by the index 6, and it is - $\frac{3}{6}x^6$ or $\frac{1}{2}x^6$,
which is the fluent of the propofed fluxion $3x^5\dot{x}$.

In like manner,

The fluent of $2ax\dot{x}$ is
The fluent of $3x^2\dot{x}$ is

The

The fluent of $4x^{\frac{1}{2}}\dot{x}$ is

The fluent of $2y^{\frac{3}{4}}\dot{y}$ is

The fluent of $az^{\frac{5}{6}}\dot{z}$ is

The fluent of $x^{\frac{1}{2}}\dot{x} + 3y^{\frac{2}{3}}\dot{y}$ is

The fluent of $x^{n-1}\dot{x}$ is

The fluent of $ny^{n-1}\dot{y}$ is

The fluent of $\frac{\dot{z}}{z^2}$, or $z^{-2}\dot{z}$ is

The fluent of $\frac{a\dot{y}}{y^n}$ is

The fluent of $(a+x)^4\dot{x}$ is

The fluent of $(a^4+y^4)y^3\dot{y}$ is

The fluent of $(a^3+z^3)^4z^2\dot{z}$ is

The fluent of $(a^n+x^n)^m x^{n-1}\dot{x}$ is

The fluent of $(a^2+y^2)^3 y\dot{y}$ is

The fluent of $\frac{z\dot{z}}{\sqrt{a^2+z^2}}$ is

The fluent of $\frac{\dot{x}}{\sqrt{a-x}}$ is

37. *When the Root under a Vinculum is a Compound Quantity; and the Index of the part or factor Without the Vinculum, increased by* 1, *is some Multiple of that Under the Vinculum:*

Put a single variable letter for the compound root; and substitute its powers and fluxion instead of those, of the same value, in the given quantity; so will it be reduced to a simpler form, to which the preceding rule can then be applied.

Thus, if the given fluxion be $\dot{F} = (a^2+x^2)^{\frac{2}{3}}x^3\dot{x}$, where 3, the index of the quantity without the vinculum, increased by 1, making 4, which is just the double of 2, the exponent of x^2 within the vinculum: therefore, putting $z = a^2 + x^2$, thence $x^2 = z - a^2$, the fluxion of which is $2x\dot{x} = \dot{z}$; hence then $x^3\dot{x} = \frac{1}{2}x^2\dot{z} = \frac{1}{2}\dot{z}(z-a^2)$, and the given fluxion $\dot{F}$, or $(a^2+x^2)^{\frac{2}{3}}x^3\dot{x}$, is $= \frac{1}{2}z^{\frac{2}{3}}\dot{z}(z-a^2)$ or $= \frac{1}{2}z^{\frac{5}{3}}\dot{z} - \frac{1}{2}a^2z^{\frac{2}{3}}\dot{z}$; and hence the fluent F is $= \frac{3}{16}z^{\frac{8}{3}} - \frac{3}{10}a^2z^{\frac{5}{3}} = 3z^{\frac{5}{3}}(\frac{1}{16}z - \frac{3}{10}a^2)$. Or, by substituting the value of z instead of it, the same fluent is $3(a^2+x^2)^{\frac{5}{3}} \times (\frac{1}{16}x^2 - \frac{3}{80}a^2)$, or $\frac{3}{16}\cdot\overline{a^2+x^2}^{\frac{5}{3}}\cdot\overline{x^2-\frac{3}{5}a^2}$.

In like manner for the following examples.

To find the fluent of $\sqrt{a + cx} \times x^3\dot{x}$.

To find the fluent of $\overline{a + cx}|^{\frac{3}{4}} x^2\dot{x}$.

To find the fluent of $(a + cx^2)^{\frac{1}{3}} \times dx^3\dot{x}$.

To find the fluent of $\dfrac{cz\dot{z}}{\sqrt{a+z}}$ or $(a + z)^{-\frac{1}{2}} cz\dot{z}$.

To find the fluent of $\dfrac{cz^{3n-1}\dot{z}}{\sqrt{a+z^n}}$ or $(a + z^n)^{-\frac{1}{2}} cz^{3n-1}\dot{z}$.

To find the fluent of $\dfrac{\dot{z}\sqrt{a^2 + z^2}}{z^6}$ or $(a^2 + z^2)^{\frac{1}{2}} z^{-6}\dot{z}$

To find the fluent of $\dfrac{\dot{x}\sqrt{a - x^n}}{x^{\frac{7}{2}n-1}}$ or $(a - x^n)^{\frac{1}{2}} x^{-\frac{7}{2}n-1}\dot{x}$.

38. *When there are ſeveral Terms, involving Two or more Variable Quantities, having the Fluxion of each Multiplied by the other Quantity or Quantities:*

Take the fluent of each term, as if there was only one variable quantity in it, namely, that whoſe fluxion is contained in it, ſuppoſing all the others to be conſtant in that term; then, if the fluents of all the terms, ſo found, be the very ſame quantity in all of them, that quantity will be the fluent of the whole. Which is the reverſe of the 5th rule for finding fluxions: Thus, if the given fluxion be $\dot{x}y + x\dot{y}$, then the fluent of $\dot{x}y$ is xy, ſuppoſing y conſtant; and the fluent of $x\dot{y}$ is alſo xy, ſuppoſing x conſtant; therefore xy is the required fluent of the given fluxion $\dot{x}y + x\dot{y}$.

In like manner,

The fluent of $\dot{x}yz + x\dot{y}z + xy\dot{z}$ is

The fluent of $2xy\dot{x} + x^2\dot{y}$ is

The fluent of $\frac{1}{2}x^{-\frac{1}{2}}\dot{x}y^2 + 2x^{\frac{1}{2}}y\dot{y}$ is

The fluent of $\dfrac{\dot{x}y - x\dot{y}}{y^2}$ or $\dfrac{\dot{x}}{y} - \dfrac{x\dot{y}}{y^2}$ is

The fluent of $\dfrac{2ax\dot{x}y^{\frac{1}{2}} - \frac{1}{2}ax^2y^{-\frac{1}{2}}\dot{y}}{y}$ or $\dfrac{2ax\dot{x}}{\sqrt{y}} - \dfrac{ax^2\dot{y}}{2y\sqrt{y}}$ is

39. When

39. *When the given Fluxional Expression is in this Form* $\frac{\dot{x}y - x\dot{y}}{y^2}$, *namely, a Fraction, including Two Quantities, being the Fluxion of the former of them drawn into the latter, minus the Fluxion of the latter drawn into the former, and divided by the Square of the latter:*

Then, the fluent is the fraction $\frac{x}{y}$, or the former quantity divided by the latter. That is,

The fluent of $\frac{\dot{x}y - x\dot{y}}{y^2}$ is $\frac{x}{y}$. And, in like manner,

The fluent of $\frac{2x\dot{x}y^2 - 2x^2y\dot{y}}{y^4}$ is $\frac{x^2}{y^2}$.

Though, indeed, the examples of this case may be performed by the foregoing one. Thus, the given fluxion $\frac{\dot{x}y - x\dot{y}}{y^2}$ reduces to $\frac{\dot{x}}{y} - \frac{x\dot{y}}{y^2}$, or $\frac{\dot{x}}{y} - x\dot{y}y^{-2}$; of which, the fluent of $\frac{\dot{x}}{y}$ is $\frac{x}{y}$, supposing y constant; and the fluent of $- x\dot{y}y^{-2}$ is also xy^{-1} or $\frac{x}{y}$, when x is constant; therefore, by that case, $\frac{x}{y}$ is the fluent of the whole $\frac{\dot{x}y - x\dot{y}}{y^2}$.

40. *When the Fluxion of a Quantity is Divided by the Quantity itself:*

Then the fluent is equal to the hyperbolic logarithm of that quantity; or, which is the same thing, the fluent is equal to 2·30258509 multiplied by the common logarithm of the same quantity.

So, the fluent of $\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$ is the hyp. log. of x.

The fluent of $\frac{2\dot{x}}{x}$ is $2 \times$ hyp. log. of x, or $=$ hyp. log. x^2.

The fluent of $\frac{a\dot{x}}{x}$ is $2 \times$ hyp. log. x, or $=$ hyp. log. of x^2.

The fluent of $\frac{\dot{x}}{a + x}$ is

The fluent of $\frac{3x^2\dot{x}}{a + x^3}$ is

 41. Many

41. *Many fluents may be found by the Direct Method thus:*

Take the fluxion again of the given fluxion, or the second fluxion of the fluent sought; into which substitute $\frac{\dot{x}^2}{x}$ for $\ddot{x}$, $\frac{\dot{y}^2}{y}$ for $\ddot{y}$, &c; that is, make x, $\dot{x}$, $\ddot{x}$, as also, y, $\dot{y}$, $\ddot{y}$, &c, to be in continual proportion, or so that $x : \dot{x} :: \dot{x} : \ddot{x}$, and $y : \dot{y} :: \dot{y} : \ddot{y}$, &c; then divide the square of the given fluxional expression by the second fluxion, just found, and the quotient will be the fluent required in many cases.

Or the same rule may be otherwise delivered thus:

In the given fluxion $\dot{F}$, write x for $\dot{x}$, y for $\dot{y}$, &c, and call the result G, taking also the fluxion of this quantity, $\dot{G}$; then make $\dot{G} : \dot{F} :: G : F$; so shall the fourth proportional F be the fluent sought in many cases.

It may be proved if this be the true fluent, by taking the fluxion of it again, which, if it agree with the proposed fluxion, will shew that the fluent is right; otherwise, it is wrong.

EXAMPLES.

EXAM. 1. Let it be required to find the fluent of $nx^{n-1}\dot{x}$.

Here $\dot{F} = nx^{n-1}\dot{x}$, write x for $\dot{x}$, then $nx^{n-1}x$ or $nx^n = G$; the fluxion of this is $\dot{G} = n^2x^{n-1}\dot{x}$; therefore $\dot{G} : \dot{F} :: G : F$, becomes $n^2x^{n-1}\dot{x} : nx^{n-1}\dot{x} :: nx^n : x^n = F$, the fluent sought.

EXAM. 2. To find the fluent of $\dot{x}y + x\dot{y}$.

Here $\dot{F} = \dot{x}y + x\dot{y}$; then, writing x for $\dot{x}$, and y for $\dot{y}$, it is $xy + xy$ or $2xy = G$; hence $\dot{G} = 2\dot{x}y + 2x\dot{y}$; then $\dot{G} : \dot{F} :: G : F$, becomes $2\dot{x}y + 2x\dot{y} : \dot{x}y + x\dot{y} :: 2xy : xy = F$, the fluent sought.

42. *To find Fluents by means of a Table of Forms of Fluxions and Fluents.*

In the following Table are contained the most usual forms of fluxions that occur in the practical solution of problems, with their corresponding fluents set opposite to them; by means of which, namely, by comparing any proposed fluxion with the corresponding form in the table, the fluent of it will be found.

Forms

Forms.	Fluxions.	Fluents.
I	$x^{n-1}\dot{x}$	$\frac{x^n}{n}$, or $\frac{1}{n}x^n$
II	$(a\pm x^n)^{m-1}x^{n-1}\dot{x}$	$\pm\frac{1}{mn}(a\pm x^n)^m$
III	$\frac{x^{mn-1}\dot{x}}{(a\pm x^n)^{m+1}}$	$\frac{1}{mna}\times\frac{x^{mn}}{(a\pm x^n)^m}$
IV	$\frac{(a\pm x^n)^{m-1}\dot{x}}{x^{mn+1}}$	$\frac{-1}{mna}\times\frac{(a\pm x^n)^m}{x^{mn}}$
V	$(my\dot{x}+nx\dot{y})\times x^{m-1}y^{n-1}$, or $\left(\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y}\right)x^my^n$	x^my^n
VI	$mx^{m-1}\dot{x}y^nz^r+nx^my^{n-1}\dot{y}z^r+rx^my^nz^{r-1}\dot{z}$, or $(m\dot{x}yz+nx\dot{y}z+rxy\dot{z})x^{m-1}y^{n-1}z^{r-1}$, or $\left(\frac{m\dot{x}}{x}+\frac{n\dot{y}}{y}+\frac{r\dot{z}}{z}\right)x^my^nz^r$,	$x^my^nz^r$
VII	$\frac{\dot{x}}{x}$ or $x^{-1}\dot{x}$	log. of x
VIII	$\frac{x^{n-1}\dot{x}}{a\pm x^n}$	$\pm\frac{1}{n}$ log. of $a\pm x^n$
IX	$\frac{x^{-1}\dot{x}}{a\pm x^n}$	$\frac{1}{na}$ log. of $\frac{x^n}{a\pm x^n}$
X	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a-x^n}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a}+\sqrt{x^n}}{\sqrt{a}-\sqrt{x^n}}$
XI	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{a+x^n}$	$\frac{2}{n\sqrt{a}}\times$ arc to tan. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}}\times$ arc to cosine $\frac{a-x^n}{a+x^n}$
XII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{\pm a+x^n}}$	$\frac{2}{n}$ log. of $\sqrt{x^n}+\sqrt{\pm a+x^n}$

Forms.	*Fluxions.*	*Fluents.*
XIII	$\frac{x^{\frac{1}{2}n-1}\dot{x}}{\sqrt{a-x^n}}$	$\frac{2}{n} \times$ arc to ſin. $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n} \times$ arc to verſ. $\frac{2x^n}{a}$
XIV	$\frac{x^{-1}\dot{x}}{\sqrt{a \pm x^n}}$	$\frac{1}{n\sqrt{a}}$ log. of $\frac{\sqrt{a \pm x^n} \mp \sqrt{a}}{\sqrt{a \pm x^n} + \sqrt{a}}$
XV	$\frac{x^{-1}\dot{x}}{\sqrt{-a + x^n}}$	$\frac{2}{n\sqrt{a}} \times$ arc to ſecant $\sqrt{\frac{x^n}{a}}$, or $\frac{1}{n\sqrt{a}} \times$ arc to coſin. $\frac{2a - x^n}{x^n}$
XVI	$\dot{x}\sqrt{dx - x^2}$	$\frac{1}{2}$ circ. ſeg. to diam. d & verſ. x
XVII	$c^{nx}\dot{x}$	$\frac{c^{nx}}{n \text{ log. } c}$
XVIII	$\dot{x}y^x \text{ log. } y + xy^{x-1}\dot{y}$	y^x

Note. The logarithms, in the above forms, are the hyperbolic ones, which are found by multiplying the common logarithms by 2·302585092994. And the arcs, whoſe ſine, or tangent, &c, are mentioned, have the radius 1, and are thoſe in the common tables of ſines, tangents, and ſecants. Alſo, the numbers m, n, &c, are to be ſome real quantities, as the forms fail when $m = 0$, or $n = 0$, &c.

The Uſe of the foregoing Table of Forms of Fluxions and Fluents.

43. In uſing the foregoing table, it is to be obſerved, that the firſt column ſerves only to ſhew the number of the form; in the ſecond column are the ſeveral forms of fluxions, which are of different kinds or claſſes; and in the third or laſt column, are the correſponding fluents.

The method of uſing the table, is this. Having any fluxion given, to find its fluent: Firſt, Compare the given fluxion with the ſeveral forms of fluxions in the ſecond column of the table, till one of the forms be found that agrees with

with it; which is done by comparing the terms of the given fluxion with the like parts of the tabular fluxion, namely, the radical quantity of the one, with that of the other; and the exponents of the variable quantities of each, both within and without the vinculum; all which, being found to agree or correſpond, will give the particular values of the general quantities in the tabular form: then ſubſtitute theſe particular values in the general or tabular form of the fluent, and the reſult will be the particular fluent of the given fluxion; after it is multiplied by any co-efficient the propoſed fluxion may have.

EXAMPLES.

EXAM. 1. To find the fluent of the fluxion $3x^{\frac{5}{3}}\dot{x}$.

This is found to agree with the firſt form. And, by comparing the fluxions, it appears that $x = x$, and $n - 1 = \frac{5}{3}$, or $n = \frac{8}{3}$; which being ſubſtituted in the tabular fluent, or $\frac{1}{n}x^n$, gives, after multiplying by 3 the co-efficients, $3 \times \frac{3}{8}x^{\frac{8}{3}}$, or $\frac{9}{8}x^{\frac{8}{3}}$, for the fluent ſought.

EXAM. 2. To find the fluent of $5x^2\dot{x}\sqrt{c^3 - x^3}$, or $5x^2\dot{x}(c^3 - x^3)^{\frac{1}{2}}$.

This fluxion, it appears, belongs to the 2d tabular form: for $a = c^3$, and $-x^n = -x^3$, and $n = 3$ under the vinculum, alſo $m - 1 = \frac{1}{2}$, or $m = \frac{3}{2}$, and the exponent ${}^{n-1}$ of x^{n-1} without the vinculum, by uſing 3 for n, is $n - 1 = 2$, which agrees with x^2 in the given fluxion: ſo that all the parts of the form are found to correſpond. Then, ſubſtituting theſe values into the general fluent, $-\frac{1}{mn}(a - x^n)^m$, it becomes $-\frac{5}{3} \times \frac{2}{3}(c^3 - x^3)^{\frac{3}{2}} = -\frac{10}{9}(c^3 - x^3)^{\frac{3}{2}}$.

EXAM. 3. To find the fluent of $\frac{x^2\dot{x}}{1 + x^3}$.

This is found to agree with the 8th form; where $\pm x^n = + x^3$ in the denominator, or $n = 3$; and the numerator x^{n-1} then becomes x^2, which agrees with the numerator in the given fluxion; alſo $a = 1$. Hence then, by ſubſtituting in the general or tabular fluent, $\frac{1}{n}$ log. of $a + x^n$, it becomes $\frac{1}{3}$ log. of $1 + x^3$.

EXAM. 4. To find the fluent of $ax^4\dot{x}$.

EXAM. 5. To find the fluent of $2(10 + x^2)^{\frac{2}{3}}x\dot{x}$.

EXAM. 6. To find the fluent of $\frac{a\dot{x}}{\overline{c^2 + x^2}|^{\frac{3}{2}}}$.

EXAM. 7. To find the fluent of $\frac{3x^2\dot{x}}{\overline{a - x}|^4}$.

EXAM. 8.

EXAM. 8. To find the fluent of $\frac{c^2 - x^2}{x^5}\dot{x}$.

EXAM. 9. To find the fluent of $\frac{1 + 3x}{2x^4}\dot{x}$.

EXAM. 10. To find the fluent of $(\frac{3\dot{x}}{x} + \frac{2\dot{y}}{y})\,x^3y^2$.

EXAM. 11. To find the fluent of $(\frac{\dot{x}}{x} + \frac{\dot{y}}{3y})\,xy^{\frac{1}{3}}$.

EXAM. 12. To find the fluent of $\frac{3\dot{x}}{ax}$ or $\frac{3}{a}x^{-1}\dot{x}$.

EXAM. 13. To find the fluent of $\frac{a\dot{x}}{3 - 2x}$.

EXAM. 14. To find the fluent of $\frac{3\dot{x}}{2x - x^2}$ or $\frac{3x^{-1}\dot{x}}{2 - x}$.

EXAM. 15. To find the fluent of $\frac{2\dot{x}}{x - 3x^3}$ or $\frac{2x^{-1}\dot{x}}{1 - 3x^2}$.

EXAM. 16. To find the fluent of $\frac{3x\dot{x}}{1 - x^4}$.

EXAM. 17. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2 - x^5}$.

EXAM. 18. To find the fluent of $\frac{2x\dot{x}}{1 + x^4}$.

EXAM. 19. To find the fluent of $\frac{ax^{\frac{3}{2}}\dot{x}}{2 + x^5}$.

EXAM. 20. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1 + x^4}}$.

EXAM. 21. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^2 - 4}}$.

EXAM. 22. To find the fluent of $\frac{3x\dot{x}}{\sqrt{1 - x^4}}$.

EXAM. 23. To find the fluent of $\frac{a\dot{x}}{\sqrt{4 - x^2}}$.

EXAM. 24. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{1 - x^2}}$.

EXAM. 25. To find the fluent of $\frac{a\dot{x}}{\sqrt{ax^2 + x^{\frac{1}{2}}}}$.

EXAM. 26. To find the fluent of $\frac{2x^{-1}\dot{x}}{\sqrt{x^2 - 1}}$.

EXAM. 27.

EXAM. 27. To find the fluent of $\frac{a\dot{x}}{\sqrt{x^{\frac{1}{2}} - ax^2}}$.

EXAM. 28. To find the fluent of $2\dot{x}\sqrt{2x - x^2}$.

EXAM. 29. To find the fluent of $a^x\dot{x}$.

EXAM. 30. To find the fluent of $3a^{2x}\dot{x}$.

EXAM. 31. To find the fluent of $3z^x\dot{x}$ log. $z + 3xz^{x-1}\dot{z}$.

EXAM. 32. To find the fluent of $(1 + x^3)\, x\dot{x}$.

EXAM. 33. To find the fluent of $(2 + x^4)\, x^{\frac{3}{2}}\dot{x}$.

EXAM. 34. To find the fluent of $x^2\dot{x}\sqrt{a^2 + x^2}$.

To find Fluents by Infinite Series.

44. When a given fluxion, whofe fluent is required, is fo complex, that it cannot be made to agree with any of the forms in the foregoing table of cafes, nor made out from the general rules before given; recourfe is then to be had to the method of infinite feries; which is thus performed:

Expand the radical or fraction, in the given fluxion, into an infinite feries of fimple terms, by the methods given for that purpofe in books of algebra; viz. either by divifion, or extraction of roots, or by the binomial theorem, &c; and multiply every term by the fluxional letter, and by fuch fimple variable factor as the given fluxional expreffion may contain. Then take the fluent of each term feparately, by the foregoing rules, connecting them all together by their proper figns; and the feries will be the fluent fought, after multiplied by any conftant factor or co-efficient which may be contained in the given fluxional expreffion.

45. It is to be noted, however, that the quantities muft be fo arranged, as that the feries produced may be a converging one, rather than diverging; and this is effected by placing the greater terms foremoft in the given fluxion. When thefe are known or conftant quantities, the infinite feries will be an afcending one; that is, the powers of the variable quantity will afcend, or increafe; but if the variable quantity be fet foremoft, the infinite feries produced will be a defcending one, or the powers of that quantity will decreafe always more and more in the fucceeding terms, or increafe in the denominators of them, which is the fame thing.

For

For example, to find the fluent of $\frac{1-x}{1+x-x^2}$.

Here, by dividing the numerator by the denominator, the proposed fluxion becomes $\dot{x}-2x\dot{x}+3x^2\dot{x}-5x^3\dot{x}+8x^4\dot{x}-$&c; then the fluents of all the terms, being taken, give - - $x-x^2+x^3-\frac{5}{4}x^4+\frac{8}{5}x^5-$&c, for the fluent sought.

Again, to find the fluent of $\dot{x}\sqrt{1-x^2}$.

Here, by extracting the root, or expanding the radical quantity $\sqrt{1-x^2}$, the given fluxion becomes - - - - $\dot{x}-\frac{1}{2}x^2\dot{x}-\frac{1}{8}x^4\dot{x}-\frac{1}{16}x^6\dot{x}-$&c. Then, the fluents of all the terms, being taken, give $x-\frac{1}{6}x^3-\frac{1}{40}x^5-\frac{1}{112}x^7-$&c, for the fluent sought.

OTHER EXAMPLES.

EXAM. 1. To find the fluent of $\frac{bx\dot{x}}{a-x}$ both in an ascending and descending series.

EXAM. 2. To find the fluent of $\frac{b\dot{x}}{a+x}$ in both series.

EXAM. 3. To find the fluent of $\frac{3\dot{x}}{(a+x)^2}$.

EXAM. 4. To find the fluent of $\frac{1-x^2+2x^4}{1+x-x^2}\dot{x}$.

EXAM. 5. Given $\dot{z}=\frac{b\dot{x}}{a^2+x^2}$, to find z.

EXAM. 6. Given $\dot{z}=\frac{a^2+x^2}{a+x}\dot{x}$, to find z.

EXAM. 7. Given $\dot{z}=3\dot{x}\sqrt{a+x}$, to find z.

EXAM. 8. Given $\dot{z}=2\dot{x}\sqrt{a^2+x^2}$, to find z.

EXAM. 9. Given $\dot{z}=4\dot{x}\sqrt{a^2-x^2}$, to find z.

EXAM. 10. Given $\dot{z}=\frac{5a\dot{x}}{\sqrt{x^2-a^2}}$, to find z.

EXAM. 11. Given $\dot{z}=2\dot{x}\sqrt[3]{a^3-x^3}$, to find z.

EXAM. 12. Given $\dot{z}=\frac{3a\dot{x}}{\sqrt{ax-xx}}$, to find z.

EXAM. 13. Given $\dot{z}=2\dot{x}\sqrt[3]{x^3+x^4+x^5}$, to find z.

EXAM. 14. Given $\dot{z}=5\dot{x}\sqrt{ax-xx}$, to find z.

To

To Correct the Fluent of any Given Fluxion.

46. The fluxion found from a given fluent, is always perfect and complete: but the fluent found from a given fluxion, is not always so; as it often wants a correction, to make it contemporaneous with that required by the problem under consideration, &c: for, the fluent of any given fluxion, as $\dot{x}$, may be either x, which is found by the rule, or it may be $x + c$, or $x - c$, that is, x plus or minus some constant quantity c; because both x and $x \pm c$ have the same fluxion $\dot{x}$, and the finding of the constant quantity c, to be added or subtracted with the fluent as found by the foregoing rules, is called *correcting* the fluent.

Now this correction is to be determined from the nature of the problem in hand, by which we come to know the relation which the fluent quantities have to each other at some certain point or time. Reduce, therefore, the general fluential equation, supposed to be found by the foregoing rules, to that point or time; then if the equation be true, it is correct; but if not, it wants a correction; and the quantity of the correction, is the difference between the two general sides of the equation when reduced to that particular point. Hence the general rule for the correction is this:

Connect the constant, but indeterminate, quantity c, with one side of the fluential equation, as determined by the foregoing rules; then, in this equation, substitute for the variable quantities, such values as they are known to have at any particular state, place, or time; and then, from that particular state of the equation, find the value of c, the constant quantity of the correction.

EXAMPLES.

47. EXAM. 1. To find the correct fluent of $\dot{z} = ax^3\dot{x}$.

The general fluent is $z = ax^4$, or $z = ax^4 + c$, taking in the correction c.

Now, if it be known that z and x begin together, or that z is $= 0$, when $x = 0$; then writing 0 for both x and z, the general equation becomes $0 = 0 + c$, or $= c$; so that, the value of c being 0, the correct fluents are $z = ax^4$.

But if z be $= 0$, when x is $= b$, any known quantity; then substituting 0 for z, and b for x, in the general equation, it becomes $0 = ab^4 + c$, and from hence we find $c = -ab^4$; which being written for c in the general fluential equation, it becomes $z = ax^4 - ab^4$, for the correct fluents.

Or,

Or, if it be known that z is $=$ ſome quantity d, when x is $=$ ſome other quantity as b; then ſubſtituting d for z, and b for x, in the general fluential equation $z = ax^4 + c$, it becomes $d = ab^4 + c$; and hence is deduced the value of the correction, namely, $c = d - ab^4$; conſequently, writing this value for c in the general equation, it becomes - - - $z = ax^4 - ab^4 + d$, for the correct equation of the fluents in this caſe.

48. And hence ariſes another eaſy and general way of correcting the fluents, which is this: In the general equation of the fluents, write the particular values of the quantities which they are known to have at any certain time or poſition; then ſubtract the ſides of the reſulting particular equation from the correſponding ſides of the general one, and the remainders will give the correct equation of the fluents ſought.

So, the general equation being $z = ax^4$;
write d for z, and b for x, then $d = ab^4$;
hence, by ſubtraction, - $z - d = ax^4 - ab^4$,
or $z = ax^4 - ab^4 + d$, the correct fluents, as before.

EXAM. 2. To find the correct fluents of $\dot{z} = 5x\dot{x}$; z being $= 0$ when x is $= a$.

EXAM. 3. To find the correct fluents of $\dot{z} = 3\dot{x}\sqrt{a + x}$; z and x being $= 0$ at the ſame time.

EXAM. 4. To find the correct fluents of $\dot{z} = \frac{3a\dot{x}}{a + x}$; ſuppoſing z and x to begin to flow together, or to be each $= 0$ at the ſame time.

EXAM. 5. To find the correct fluents of $\dot{z} = \frac{2\dot{x}}{a^2 + x^2}$; ſuppoſing z and x to begin together.

OF MAXIMA AND MINIMA; OR, THE GREATEST AND LEAST MAGNITUDE OF VARIABLE OR FLOWING QUANTITIES.

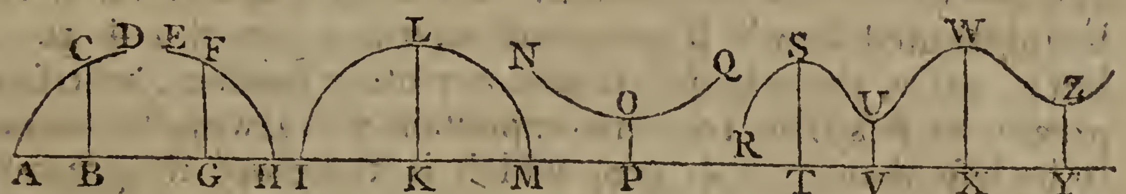
49. MAXIMUM, denotes the greateſt ſtate or quantity attainable in any given caſe, or the greateſt value of a variable quantity: by which it ſtands oppoſed to Minimum, which is the leaſt poſſible quantity in any caſe.

Thus,

Thus, the expreſſion or ſum $a^2 + bx$, evidently increaſes as x, or the term bx, increaſes; therefore the given expreſſion will be the greateſt, or a maximum, when x is the greateſt, or infinite; and the ſame expreſſion will be a minimum, or the leaſt, when x is the leaſt, or nothing.

Again, in the algebraic expreſſion $a^2 - bx$, where a and b denote conſtant or invariable quantities, and x a flowing or variable one. Now, it is evident that the value of this remainder or difference, $a^2 - bx$, will increaſe, as the term bx, or as x, decreaſes; therefore the former will be the greateſt, when the latter is the ſmalleſt; that is, $a^2 - bx$ is a maximum, when x is the leaſt, or nothing at all; and the difference is the leaſt, when x is the greateſt.

50. Some variable quantities increaſe continually; and ſo have no maximum, but what is infinite. Others again decreaſe continually; and ſo have no minimum, but what is of no magnitude, or nothing. But, on the other hand, ſome variable quantities increaſe only to a certain finite magnitude, called their Maximum, or greateſt ſtate, and after that they decreaſe again. While others decreaſe to a certain finite magnitude, called their Minimum, or leaſt ſtate, and afterwards increaſe again. And laſtly, ſome quantities have ſeveral maxima and minima.



Thus, for example, the ordinate BC of the parabola, or ſuch-like curve, flowing along the axis AB from the vertex A, continually increaſes, and has no limit or maximum. And the ordinate GF of the curve EFH, flowing from E towards H, continually decreaſes to nothing when it arrives at the point H. But in the circle ILM, the ordinate only increaſes to a certain magnitude, namely the radius, when it arrives at the middle as at KL, which is its maximum; and after that it decreaſes again to nothing, at the point M. And in the curve NOQ, the ordinate decreaſes only to the poſition OP, where it is leaſt, or a minimum; and after that it continually increaſes towards Q. But in the curve RSU, &c, the ordinates have ſeveral maxima, as ST, WX, and ſeveral minima, as VU, YZ, &c.

51. Now,

51. Now, becaufe the fluxion of a variable quantity, is the rate of its increafe or decreafe; and becaufe the maximum or minimum of a quantity neither increafes nor decreafes, at thofe points or ftates; therefore fuch maximum or minimum has no fluxion, or the fluxion is then equal to nothing. From which we have the following rule.

To find the Maximum or Minimum.

52. From the nature of the queftion or problem, find an algebraical expreffion for the value, or general ftate, of the quantity whofe maximum or minimum is required; then take the fluxion of that expreffion, and put it equal to nothing; from which equation, by dividing by, or leaving out, the fluxional letter and other common quantities, and performing other proper reductions, as in common algebra, the value of the unknown quantity will be obtained, determining the point of the maximum or minimum.

So, if it be required to find the maximum ftate of the compound expreffion $100x - 5x^2 \pm c$, or the value of x when $100x - 5x^2 \pm c$ is a maximum. The fluxion of this expreffion is $100\dot{x} - 10x\dot{x} = 0$; which being made $= 0$, and divided by $10\dot{x}$, the equation is $10 - x = 0$; and hence $x = 10$. That is, the value of x is 10, when the expreffion $100x - 5x^2 \pm c$ is the greateft. As is eafily tried: for if 10 be fubftituted for x in that expreffion, it becomes $\pm c + 500$: but if, for x, there be fubftituted any other number, whether greater or lefs than 10; that expreffion will always be found to be lefs than $\pm c + 500$, which is therefore its greateft poffible value, or its maximum.

53. It is evident, that if a maximum or minimum be any way compounded with, or operated upon, by a given conftant quantity, the refult will ftill be a maximum or minimum. That is, if a maximum or minimum be increafed, or decreafed, or multiplied, or divided, by a given quantity, or any given power or root of it be taken; the refult will ftill be a maximum or minimum. Thus, if x be a maximum or minimum, then alfo is $x + a$, or $x - a$, or ax, or $\frac{x}{a}$, or x^2, or $\sqrt[3]{x}$, ftill a maximum or minimum. Alfo, the logarithm of the fame will be a maximum or a minimum. And therefore, if any propofed maximum or minimum can be made fimpler by performing any of thefe operations, it is better to do fo, before the expreffion is put into fluxions.

54. When

54. When the expreffion for a maximum or minimum contains feveral variable letters or quantities; take the fluxion of it as often as there are variable letters; fuppofing firft one of them only to flow, and the reft to be conftant; then another only to flow, and the reft conftant; and fo on for all of them: then putting each of thefe fluxions $= 0$, there will be as many equations as unknown letters, from which thefe may be all determined. For the fluxion of the expreffion muft be equal to nothing in each of thefe cafes; otherwife the expreffion might become greater or lefs, without altering the values of the other letters, which are confidered as conftant.

So, if it be required to find the values of x and y when $4x^2 - xy + 2y$ is a minimum. Then we have,

First - $8x\dot{x} - \dot{x}y = 0$, and $8x - y = 0$, or $y = 8x$.

Secondly, $2\dot{y} - x\dot{y} = 0$, and $2 - x = 0$, or $x = 2$.

And hence y or $8x = 16$.

55. *To find whether a propofed quantity admits of a Maximum or a Minimum.*

Every algebraic expreffion does not admit of a maximum or minimum, properly fo called; for it may either increafe continually to infinity, or decreafe continually to nothing; and, in both thefe cafes, there is neither a proper maximum nor minimum; for the true maximum is that finite value to which an expreffion increafes, after which it decreafes again: and the minimum is that finite value to which the expreffion decreafes, and after that it increafes again. Therefore, when the expreffion admits of a maximum, its fluxion is pofitive before that point, and negative after it; but when it admits of a minimum, its fluxion is negative before, and pofitive after it. Hence, taking the fluxion of the expreffion a little before the fluxion is equal to nothing, and again a little after the fame; if the former fluxion be pofitive, and the latter negative, the middle ftate is a maximum; but if the former fluxion be negative, and the latter pofitive, the middle ftate is a minimum.

So, if we would find the quantity $ax - x^2$ a maximum or minimum; make its fluxion equal to nothing, that is, $a\dot{x} - 2x\dot{x} = 0$, or $(a - 2x)\dot{x} = 0$; dividing by $\dot{x}$, gives $a - 2x = 0$, or $x = \frac{1}{2}a$ at that ftate. Now, if in the fluxion $(a - 2x)\dot{x}$, the value of x be taken rather lefs than its true value, $\frac{1}{2}a$, that fluxion will evidently be pofitive: but if x be taken fomewhat greater than $\frac{1}{2}a$, the value of $a - 2x$, and confequently of the fluxion, is as evidently negative. Therefore, the fluxion of $ax - x^2$ being pofitive before, and negative

gative after the ſtate when its fluxion is $= 0$, it follows that at this ſtate the expreſſion is not a minimum, but a maximum.

Again, taking the expreſſion $x^3 - ax^2$, its fluxion $3x^2\dot{x} - 2ax\dot{x} = (3x - 2a)x\dot{x} = 0$; this divided by $x\dot{x}$ gives $3x - 2a = 0$; and $x = \frac{2}{3}a$, its true value when the fluxion of $x^3 - ax^2$ is equal to nothing. But now to know whether the given expreſſion be a maximum or a minimum at that time, take x a little leſs than $\frac{2}{3}a$ in the value of the fluxion $(3x - 2a)\,x\dot{x}$, and this will evidently be negative; and again, taking x a little more than $\frac{2}{3}a$, the value of $3x - 2a$, or of the fluxion, is as evidently poſitive. Therefore the fluxion of $x^3 - ax^2$ being negative before that fluxion is $= 0$, and poſitive after it; it follows that in this ſtate the quantity $x^3 - ax^2$ admits of a minimum, but not of a maximum.

56. Some Examples for Practice.

Exam. 1. To divide a line, or any other given quantity a, into two parts, ſo that their rectangle or product may be the greateſt poſſible.

Exam. 2. To divide the given quantity a into two parts ſuch, that the product of the m power of one, by the n power of the other, may be a maximum.

Exam. 3. To divide the given quantity a into three parts ſuch, that the continual product of them all may be a maximum.

Exam. 4. To divide the given quantity a into three parts ſuch, that the continual product of the 1ſt, the ſquare of the 2d, and the cube of the 3d, may be a maximum.

Exam. 5. To determine a fraction ſuch, that the difference between its m power and n power ſhall be the greateſt poſſible.

Exam. 6. To divide the number 80 into two ſuch parts, x and y, that $2x^2 + xy + 3y^2$ may be a minimum.

Exam. 7. To find the greateſt rectangle that can be inſcribed in a given right-angled triangle.

Exam. 8. To find the greateſt rectangle that can be inſcribed in the quadrant of a given circle.

Exam. 9. To find the leaſt right-angled triangle that can circumſcribe the quadrant of a given circle.

Exam. 10. To find the greateſt rectangle inſcribed in, and the leaſt iſoſceles triangle circumſcribed about, a given ſemi-ellipſe.

Exam. 11.

Exam. 11. To determine the ſame for a given parabola.

Exam. 12. To determine the ſame for a given hyperbola.

Exam. 13. To inſcribe the greateſt cylinder in a given cone; or to cut the greateſt cylinder out of a given cone.

Exam. 14. To determine the dimenſions of a rectangular ciſtern, capable of containing a given quantity a of water, ſo as to be lined with lead at the leaſt poſſible expence.

Exam. 15. Required the dimenſions of a cylindrical tankard, to hold one quart of ale meaſure, that can be made of the leaſt poſſible quantity of ſilver, of a given thickneſs.

Exam. 16. To cut the greateſt parabola from a given cone.

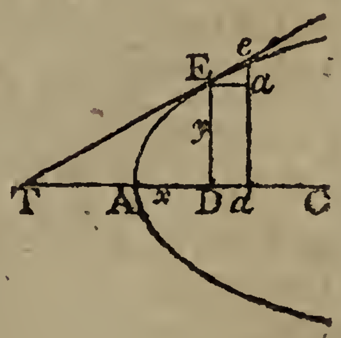
Exam. 17. To cut the greateſt ellipſe from a given cone.

Exam. 18. To find the value of x when x^x is a minimum.

The METHOD of TANGENTS; Or, To draw TANGENTS to CURVES.

THE Method of Tangents, is a method of determining the quantity of the tangent and ſubtangent of any algebraic curve; the equation of the curve being given. Or, *vice verſa*, the nature of the curve, from the tangent given.

57. If AE be any curve, and E be any point in it, to which it is required to draw a tangent TE. Draw the ordinate ED: then if we can determine the ſubtangent TD, limited between the ordinate and tangent, in the axis produced, by joining the points T, E, the line TE will be the tangent ſought.



58. Let dae be another ordinate, indefinitely near to DE, meeting the curve, or tangent produced, in e; and let Ea be parallel to the axis AD. Then is the elementary triangle Eea ſimilar to the triangle TDE;

 and

and therefore - $ea : aE :: ED : DT$.
But - - $ea : aE ::$ flux. ED : flux. AD.
Therefore - flux. ED : flux. AD :: DE : DT.
That is, - $\dot{y} : \dot{x} :: y : \frac{y\dot{x}}{\dot{y}} = DT$,

which is therefore the value of the ſubtangent ſought; where x is the abſciſs AD, and y the ordinate DE.

Hence we have this general rule.

GENERAL RULE.

59. By means of the given equation of the curve, when put into fluxions, find the value of either $\dot{x}$ or $\dot{y}$, or of $\frac{\dot{x}}{\dot{y}}$; which value ſubſtitute for it in the expreſſion $DT = \frac{y\dot{x}}{\dot{y}}$, and, when reduced to its ſimpleſt terms, it will be the value of the ſubtangent ſought.

EXAMPLES.

EXAM. 1. Let the propoſed curve be that which is defined, or expreſſed, by the equation $ax^2 + xy^2 - y^3 = 0$.

Here the fluxion of the equation of the curve is $2ax\dot{x} + y^2\dot{x} + 2xy\dot{y} - 3y^2\dot{y} = 0$; then, by tranſpoſition, $2ax\dot{x} + y^2\dot{x} = 3y^2\dot{y} - 2xy\dot{y}$; and hence, by diviſion, $\frac{\dot{x}}{\dot{y}} = \frac{3y^2 - 2xy}{2ax + y^2}$; conſequently $\frac{y\dot{x}}{\dot{y}} = \frac{3y^3 - 2xy^2}{2ax + y^2}$, which is the value of the ſubtangent TD ſought.

EXAM. 2. To draw a tangent to a circle; the equation of which is $ax - x^2 = y^2$; where x is the abſciſs, y the ordinate, and a the diameter.

EXAM. 3. To draw a tangent to a parabola; its equation being $ax = y^2$; where a denotes the parameter of the axis.

EXAM. 4. To draw a tangent to an ellipſe; its equation being $c^2(ax - x^2) = a^2y^2$; where a and c are the two axes.

EXAM. 5. To draw a tangent to an hyperbola; its equation being $c^2(ax + x^2) = a^2y^2$; where a and c are the two axes.

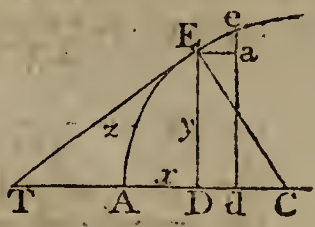
EXAM. 6. To draw a tangent to the hyperbola referred to the aſymptote as an axis; its equation being $xy = a^2$; where a^2 denotes the rectangle of the abſciſs and ordinate anſwering to the vertex of the curve.

OF

Of RECTIFICATIONS;
Or, To find the LENGTHS of CURVE LINES.

RECTIFICATION, is the finding the length of a curve line, or finding a right line equal to a propofed curve.

60. By art. 10 it appears, that the elementary triangle Eae, formed by the increments of the abfcifs, ordinate, and curve, is a right-angled triangle, of which the increment of the curve is the hypothenufe; and therefore the fquare of the latter is equal to the fum of the fquares of the two former; that is, $Ee^2 = Ea^2 + ae^2$. Or, fubftituting, for the increments, their proportional fluxions, it is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y}$, or $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; where z denotes any curve line AE, x its abfcifs AD, and y its ordinate DE. Hence this rule.



RULE.

61. From the given equation of the curve put into fluxions, find the value of $\dot{x}^2$ or $\dot{y}^2$, which value fubftitute inftead of it in the equation $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$; then the fluents, being taken, will give the value of z, or the length of the curve, in terms of the abfcifs or ordinate.

EXAMPLES.

Exam. 1. To find the length of the arc of a circle, in terms both of the fine, verfed fine, tangent, and fecant.

The equation of the circle may be expreffed in terms of the radius, and either the fine, or the verfed fine, or tangent, or fecant, &c, of an arc. Let, therefore, the radius of the circle be CA or CE $= r$, the verfed fine AD (of the arc AE) $= x$, the right fine DE $= y$, the tangent TE $= t$, and the fecant CT $= s$; then, by the nature of the circle, there arife thefe equations, viz.

$$y^2 = 2rx - x^2 = \frac{r^2t^2}{r^2 + t^2} = \frac{s^2 - r^2}{s^2} r^2.$$

Then, by means of the fluxions of thefe equations, with the general fluxional equation $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$, are obtained the following fluxional forms, for the fluxion of the curve; the fluent of any one of which will be the curve itfelf; viz.

$$\dot{z} = \frac{r\dot{x}}{\sqrt{2rx - xx}} = \frac{r\dot{y}}{\sqrt{r^2 - y^2}} = \frac{r^2\dot{t}}{r^2 + t^2} = \frac{r^2\dot{s}}{\sqrt{s^2 - r^2}}.$$

 Hence

Hence the value of the curve, from the fluent of each of these, gives the four following forms, in series, viz. putting $d = 2r$ the diameter, the curve is z

$$= (1 + \frac{x}{2.3d} + \frac{3x^2}{2.4.5d^2} + \frac{3.5x^3}{2.4.6.7d^3} + \&c) \sqrt{dx},$$

$$= (1 + \frac{y^2}{2.3r^2} + \frac{3y^4}{2.4.5r^4} + \frac{3.5y^6}{2.4.6.7r^6} + \&c)\, y,$$

$$= (1 - \frac{t^2}{3r^2} + \frac{t^4}{5r^4} - \frac{t^6}{7r^6} + \frac{t^8}{9r^8} - \&c)\, t,$$

$$= (\frac{s-r}{s} + \frac{s^3 - r^3}{2.3s^3} + \frac{3(s^5 - r^5)}{2.4.5s^5} + \&c)\, r.$$

Now, it is evident that the simplest of these series, is the third in order, or that which is expressed in terms of the tangent. That form will therefore be the fittest to calculate an example by in numbers. And, for this purpose, it will be convenient to assume some arc whose tangent, or at least the square of it, is known to be some small simple number. Now, the arc of 45 degrees, it is known, has its tangent equal to the radius; and therefore, taking the radius $r = 1$, and consequently the tangent of 45°, or t, $= 1$ also, in this case the arc of 45° to the radius 1, or the arc of the quadrant to the diameter 1, will be equal to the infinite series $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \&c.$

But as this series converges very slowly, it will be proper to take some smaller arc, that the series may converge faster; such as the arc of 30 degrees, the tangent of which is $= \sqrt{\frac{1}{3}}$, or its square $t^2 = \frac{1}{3}$: which being substituted in the series, the length of the arc of 30° comes out - - - -

$(1 - \frac{1}{3 \cdot 3} + \frac{1}{5 \cdot 3^2} - \frac{1}{7 \cdot 3^3} + \frac{1}{9 \cdot 3^4} - \&c)\sqrt{\frac{1}{3}}$. Hence, to compute these terms in decimal numbers, after the first, the succeeding terms will be found by dividing always by 3, and these quotients again by the absolute numbers 3, 5, 7, 9, &c; and lastly, adding every other term together, into two sums, the one the sum of the positive terms, and the other the sum of the negative ones; then lastly, the one sum taken from the other, leaves the length of the arc of 30 degrees; which being the 12th part of the whole circumference when the radius is 1, or the 6th part when the diameter is 1, consequently 6 times that arc will be the length of the whole circumference to the diameter 1. Therefore, multiplying the first term $\sqrt{\frac{1}{3}}$ by 6, the product is $\sqrt{12} = 3.4641016$; and hence the operation will be conveniently made as follows:

+ Terms.

		+ Terms.	— Terms.
1)	3·4641016 (	3·4641016	
3)	1·1547005 (		0·3849002
5)	3849002 (	769800	
7)	1283001 (		183286
9)	427667 (	47519	
11)	142556 (		12960
13)	47519 (	3655	
15)	15840 (		1056
17)	5280 (	311	
19)	1760 (		93
21)	587 (	28	
23)	196 (		8
25)	65 (	3	
27)	22 (		1
		+ 3·5462332	—0·4046406
		— 0·4046406	

So that at laſt 3·1415926 is the whole circumference to the diameter 1.

Exam. 2. To find the length of a parabola.

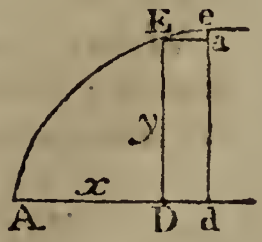
Exam. 3. To find the length of the ſemicubical parabola, whoſe equation is $ax^2 = y^3$.

Exam. 4. To find the length of an elliptical curve.

Exam. 5. To find the length of an hyperbolic curve.

Of QUADRATURES; or, Finding the AREAS of CURVES.

62. THE Quadrature of Curves, is the meaſuring their areas, or finding a ſquare, or other right-lined ſpace, equal to a propoſed curvilineal one.



By art. 9 it appears, that any flowing quantity being drawn into the fluxion of the line along which it flows, or in the direction of its motion, there is produced the fluxion of the quantity generated by the flowing. That is, Dd × DE or $y\dot{x}$ is the fluxion of the area ADE. Hence this rule.

RULE.

RULE.

63. From the given equation of the curve, find the value either of $\dot{x}$ or of y; which value ſubſtitute inſtead of it in the expreſſion $y\dot{x}$; then the fluent of that expreſſion, being taken, will be the area of the curve ſought.

EXAMPLES.

EXAM. 1. To find the area of the common parabola.

The equation of the parabola being $ax = y^2$; where a is the parameter, x the abſciſs AD, or part of the axis, and y the ordinate DE.

From the equation of the curve is found $y = \sqrt{ax}$. This ſubſtituted in the general fluxion of the area $y\dot{x}$, gives $\dot{x}\sqrt{ax}$ or $a^{\frac{1}{2}}x^{\frac{1}{2}}\dot{x}$ is the fluxion of the parabolic area; and the fluent of this, or $\frac{2}{3}a^{\frac{1}{2}}x^{\frac{3}{2}} = \frac{2}{3}x\sqrt{ax} = \frac{2}{3}xy$, is the area of the parabola ADE, and which is therefore equal to $\frac{2}{3}$ of its circumſcribing rectangle.

EXAM. 2. To ſquare the circle, or find its area.

The equation of the circle being $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$, where a is the diameter; by ſubſtitution, the general fluxion of the area $y\dot{x}$, becomes $\dot{x}\sqrt{ax - x^2}$, for the fluxion of the circular area. But as the fluent of this cannot be found in finite terms, the quantity $\sqrt{ax - x^2}$ is thrown into a ſeries, by extracting the root, and then the fluxion of the area becomes

$$\dot{x}\sqrt{ax} \times (1 - \frac{x}{2a} - \frac{x^2}{2.4a^2} - \frac{1.3x^3}{2.4.6a^3} - \frac{1.3.5x^4}{2.4.6.8a^4} - \&c);$$

then the fluent of every term being taken, it gives

$$x\sqrt{ax} \times (\frac{2}{3} - \frac{1.x}{5a} - \frac{1.x^2}{4.7a^2} - \frac{1.3x^3}{4.6.9a^3} - \&c)$$

for the general expreſſion of the ſemiſegment ADE.

When the point D arrives at the extremity of the diameter, then the ſpace becomes a ſemicircle, and $x = a$; and then the ſeries above becomes barely

$$a^2(\frac{2}{3} - \frac{1}{5} - \frac{1}{4.7} - \frac{1.3}{4.6.9} - \&c)$$

for the area of the ſemicircle whoſe diameter is a.

EXAM. 3.

EXAM. 3. To find the area of any parabola, whose equation is $a^m x^n = y^{m+n}$.

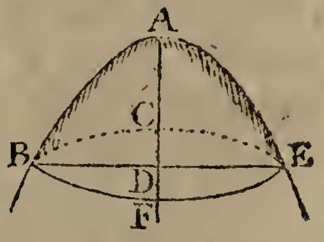
EXAM. 4. To find the area of an ellipse.

EXAM. 5. To find the area of an hyperbola.

EXAM. 6. To find the area between the curve and asymptote of an hyperbola.

EXAM. 7. To find the like area in any other hyperbola whose general equation is $x^m y^n = a^{m+n}$.

TO FIND THE SURFACES OF SOLIDS.



64. IN the solid formed by the rotation of any curve about its axis, the surface may be considered as generated by the circumference of an expanding circle, moving perpendicularly along the axis, but the expanding circumference moving along the arc or curve of the solid. Therefore, as the fluxion of any generated quantity, is produced, by drawing the generating quantity into the fluxion of the line or direction in which it moves, the fluxion of the surface will be found by drawing the circumference of the generating circle into the fluxion of the curve. That is, the fluxion of the surface BAE, is equal to $\dot{AE}$ drawn into the circumference BCEF, whose radius is the ordinate DE.

65. But, if c be $= 3{\cdot}1416$, the circumference of a circle whose diameter is 1, $x =$ AD the abscis, $y =$ DE the ordinate, and $z =$ AE the curve; then $2y =$ the diameter BE, and $2cy =$ the circumference BCEF; also, $\dot{AE} = \dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2}$: therefore $2cy\dot{z}$ or $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$ is the fluxion of the surface. And consequently if, from the given equation of the curve, the value of $\dot{x}$ or $\dot{y}$ be found, and substituted in this expression $2cy\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluent of the expression, being then taken, will be the surface of the solid required.

EXAMPLES.

EXAM. 1. To find the surface of a sphere, or of any segment.

In

In this case, AE is a circular arc, whose equation is $y^2 = ax - x^2$, or $y = \sqrt{ax - x^2}$.

The fluxion of this gives $\dot{y} = \frac{a - 2x}{2\sqrt{ax - x^2}}\dot{x} = \frac{a - 2x}{2y}\dot{x}$;

hence $\dot{y}^2 = \frac{a^2 - 4ax + 4x^2}{4y^2}\dot{x}^2 = \frac{a^2 - 4y^2}{4y^2}\dot{x}^2$;

conseq. $\dot{x}^2 + \dot{y}^2 = \frac{a^2\dot{x}^2}{4y^2}$, and $\dot{z} = \sqrt{\dot{x}^2 + \dot{y}^2} = \frac{a\dot{x}}{2y}$.

This value of $\dot{z}$, the fluxion of a circular arc, may be found more easily thus: In the fig. to art. 60, the two triangles EDC, E*ae* are equiangular, being each of them equiangular to the triangle ETC: conseq. ED : EC :: E*a* : E*e*, that is, $y : \frac{1}{2}a :: \dot{x} : \dot{z} = \frac{a\dot{x}}{2y}$, the same as before.

The value of $\dot{z}$ being found, by substitution is obtained $2cy\dot{z} = ac\dot{x}$ for the fluxion of the spherical surface, generated by the circular arc in revolving about the diameter AD. And the fluent of this gives acx for the said surface of the spherical segment BAE.

But ac is equal to the whole circumference of the generating circle; and therefore it follows, that the surface of any spherical segment, is equal to the said circumference of the generating circle, drawn into x or AD, the height of the segment.

Also, when x or AD becomes equal to the whole diameter a, the expression acx becomes aca or ca^2, or 4 times the area of the generating circle, for the surface of the whole sphere.

And these agree with the rules before found in Mensuration of Solids.

EXAM. 2. To find the surface of a spheroid.

EXAM. 3. To find the surface of a paraboloid.

EXAM. 4. To find the surface of an hyperboloid.

TO FIND THE CONTENTS OF SOLIDS.

66. ANY solid which is formed by the revolution of a curve about its axis (see last fig.), may also be conceived to be generated by the motion of the plane of an expanding circle, moving perpendicularly along the axis. And there-

fore

fore the area of that circle being drawn into the fluxion of the axis, will produce the fluxion of the solid. That is, $\dot{A}D \times$ area of the circle BCF, whose radius is DE, or diameter BE, is the fluxion of the solid, by art. 9.

67. Hence, if $AD = x$, $DE = y$, $c = 3{\cdot}1416$; because cy^2 is equal to the area of the circle BCF; therefore $cy^2\dot{x}$ is the fluxion of the solid. And consequently if, from the given equation of the curve, the value of either y^2 or $\dot{x}$ be found, and that value substituted for it in the expression by $cy^2\dot{x}$, the fluent of the resulting quantity, being taken, will be the solidity of the figure proposed.

EXAMPLES.

EXAM. 1. To find the solidity of a sphere, or any segment.

The equation to the generating circle being $y^2 = ax - x^2$, where a denotes the diameter, by substitution, the general fluxion of the solid, $cy^2\dot{x}$, becomes $cax\dot{x} - cx^2\dot{x}$, the fluent of which gives $\frac{1}{2}cax^2 - \frac{1}{3}cx^3$, or $\frac{1}{6}cx^2(3a - 2x)$, for the solid content of the spherical segment BAE, whose height AD is x.

When the segment becomes equal to the whole sphere, then $x = a$, and the above expression for the solidity, becomes $\frac{1}{6}ca^3$ for the solid content of the whole sphere.

And these deductions agree with the rules before given and demonstrated in the Mensuration of Solids.

EXAM. 2. To find the solidity of a spheroid.

EXAM. 3. To find the solidity of a paraboloid.

EXAM. 4. To find the solidity of an hyperboloid.

TO FIND LOGARITHMS.

68. IT has been proved, art. 23, that the fluxion of the hyperbolic logarithm of a quantity, is equal to the fluxion of the quantity divided by the same quantity. Therefore, when any quantity is proposed, to find its logarithm; take the fluxion of that quantity, and divide it by the same quantity; then take the fluent of the quotient, either in a series or otherwise, and it will be the logarithm sought; when corrected as usual, if need be; that is, the hyperbolic logarithm.

69. But, for any other logarithm, multiply the hyperbolic logarithm, above found, by the modulus of the system, for the logarithm sought.

Note,

Note, The modulus of the hyperbolic logarithms, is 1; and the modulus of the common logarithms, is ·43429448190, &c; and, in general, the modulus of any system, is equal to the logarithm of 10 in that system divided by the number 2·3025850929940, &c, which is the hyp. log. of 10.

EXAM. 1. To find the log. of $\frac{a+x}{a}$.

Denoting any proposed number z, whose logarithm is required to be found, by the compound expression - - $\frac{a+x}{a}$, the fluxion of the number $\dot{z}$, is $\frac{\dot{x}}{a}$, and the fluxion of the log. $\frac{\dot{z}}{z} = \frac{\dot{x}}{a+x} = \frac{\dot{x}}{a} - \frac{x\dot{x}}{a^2} + \frac{x^2\dot{x}}{a^3} - \frac{x^3\dot{x}}{a^4} +$ &c.

Then the fluents of these terms give the logarithm of z or logarithm of - - $\frac{a+x}{a} = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

writing $-x$ for x, gives log. $\frac{a-x}{a} = -\frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4}$ &c.

Mult. these numbs. and adding their logs. gives } log. $\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5}$ &c.

Also, because $\frac{a}{a+x} = 1 \div \frac{a+x}{a}$, or log. $\frac{a}{a+x} = 0 - \log. \frac{a+x}{a}$.

therefore log. of $\frac{a}{a+x}$ is $-\frac{x}{a} + \frac{x^2}{2a^2} - \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

and the log. of $\frac{a}{a-x}$ is $+\frac{x}{a} + \frac{x^2}{2a^2} + \frac{x^3}{3a^3} + \frac{x^4}{4a^4}$ &c,

the prod. gives log. $\frac{a^2}{a^2-x^2} = \frac{2x^2}{2a^2} + \frac{2x^4}{4a^4} + \frac{2x^6}{6a^6} +$ &c.

Now, for an example in numbers, suppose it were required to compute the common logarithm of the number 2. This will be best done by the series,

$$\log. \text{ of } \frac{a+x}{a-x} = 2m \times \left(\frac{x}{a} + \frac{x^3}{3a^3} + \frac{x^5}{5a^5} + \frac{x^7}{7a^7} \text{ \&c}\right).$$

Making $\frac{a+x}{a-x} = 2$, gives $a = 3x$; conseq. $\frac{x}{a} = \frac{1}{3}$, and $\frac{x^2}{a^2} = \frac{1}{9}$, which is the constant factor for every succeeding term; also, $2m = 2 \times {\cdot}43429448190 = {\cdot}868588964$; therefore the calculation will be conveniently made, by first dividing this number by 3, then the quotients successively by 9, and last-

ly

ly theſe quotients in order by the reſpective numbers 1, 3, 5, 7, 9, &c, and after that, adding all the terms together, as follows:

3)	·868588964				
9)	289529654	1)	·289529654	(	·289529654
9)	32169962	3)	32169962	(	10723321
9)	3574440	5)	3574440	(	714888
9)	397160	7)	397160	(	56737
9)	44129	9)	44129	(	4903
9)	4903	11)	4903	(	446
9)	545	13)	545	(	42
9)	61	15)	61	(	4

Sum of the terms gives log. 2 = ·301029995

EXAM. 2. To find the log. of $\frac{a+x}{b}$.

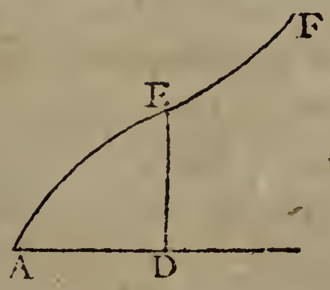
EXAM. 3. To find the log of $a - x$.

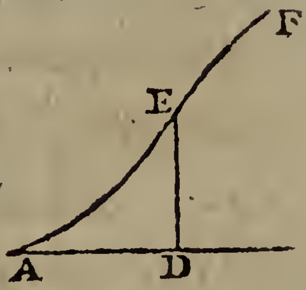
EXAM. 4. To find the log. of $\frac{a+x}{a-x}$.

EXAM. 5. To find the log. of $\frac{a-x}{a+x}$.

TO FIND THE POINTS OF INFLEXION, OR OF CONTRARY FLEXURE, IN CURVES.

70. THE Point of Inflexion in a curve, is that point of it which ſeparates the concave from the convex part, lying between the two; or where the curve changes from concave to convex, or from convex to concave, on the ſame ſide of the curve. Such as the point E in the annexed figures; where the former of the two is concave towards the axis AD, from A to E, but convex from E to F; and, on the contrary, the latter figure is convex from A to E, and concave from E to F.





71. From

71. From the nature of curvature, as has been remarked before at art. 28, it is evident that, when a curve is concave towards an axis, then the fluxion of the ordinate decreafes, or is in a decreafing ratio, with regard to the fluxion of the abfcifs; but, on the contrary, that it increafes, or is in an increafing ratio to the fluxion of the abfcifs, when the curve is convex towards the axis; and confequently thofe two fluxions are in a conftant ratio at the point of inflexion, where the curve is neither convex nor concave; that is, $\dot{x}$ is to $\dot{y}$ in a conftant ratio, or $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is a conftant quantity. But conftant quantities have no fluxion, or their fluxion is equal to nothing; fo that, in this cafe, the fluxion of $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$ is equal to nothing. And hence we have this general rule:

72. Put the given equation of the curve into fluxions; from which find either $\frac{\dot{y}}{\dot{x}}$ or $\frac{\dot{x}}{\dot{y}}$. Then take the fluxion of this ratio, or fraction, and put it equal o or nothing; and from this laft equation find alfo the value of the fame $\frac{\dot{x}}{\dot{y}}$ or $\frac{\dot{y}}{\dot{x}}$. Then put this latter value equal to the former, which will form an equation, from whence, and the firft given equation, of the curve, x and y will be determined, being the abfcifs and ordinate anfwering to the point of inflexion in the curve, as required.

EXAMPLES.

EXAM. 1. To find the point of inflexion in the curve whofe equation is $ax^2 = a^2y + x^2y$.

This equation in fluxions is $2ax\dot{x} = a^2\dot{y} + 2xy\dot{x} + x^2\dot{y}$, which gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{2ax - 2xy}$. Then the fluxion of this quantity made $= 0$, gives $2x\dot{x}\,(ax - xy) = (a^2 + x^2) \times (a\dot{x} - \dot{x}y - x\dot{y})$; and this again gives $\frac{\dot{x}}{\dot{y}} = \frac{a^2 + x^2}{a^2 - x^2} \times \frac{x}{a - y}$.

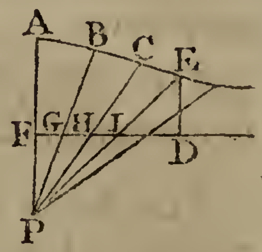
Laftly, this value of $\frac{\dot{x}}{\dot{y}}$ being put equal the former, gives $\frac{a^2 + x^2}{a^2 - x^2} \cdot \frac{x}{a - y} = \frac{a^2 + x^2}{2x} \cdot \frac{1}{a - y}$; and hence $2x^2 = a^2 - x^2$, or $3x^2 = a^2$, and $x = a\sqrt{\tfrac{1}{3}}$, the abfcifs.

Hence

Hence alſo, from the original equation,
$y = \frac{ax^2}{a^2 + x^2} = \frac{\frac{1}{3}a^3}{\frac{4}{3}a^2} = \frac{1}{4}a$, the ordinate to the point of inflexion ſought.

EXAM. 2. To find the point of inflexion in a curve defined by the equation $ay = a\sqrt{ax + xx}$.

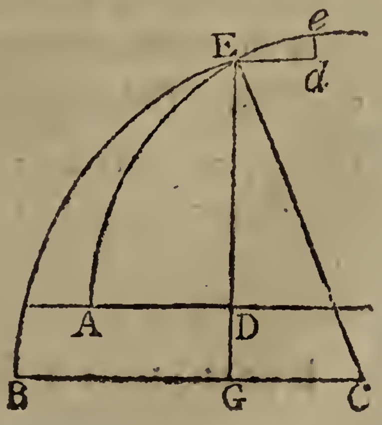
EXAM. 3. To find the point of inflexion in a curve defined by the equation $ay^2 = a^2x + x^3$.



EXAM. 4. To find the point of inflexion in the Conchoid of Nicomedes, which is generated or conſtructed in this manner: From a fixed point P, which is called the pole of the conchoid, draw any number of right lines PA, PB, PC, PE, &c, cutting the given line FD in the points F, G, H, I, &c: then make the diſtances FA, GB, HC, IE, &c, equal to each other, and equal to a given line; and the curve line ABCE, &c, will be the conchoid; a curve ſo called by its inventor Nicomedes,

TO FIND THE RADIUS OF CURVATURE OF CURVES.

73. THE Curvature of a Circle is conſtant, or the ſame in every point of it, and its radius is the radius of curvature. But the caſe is different in other curves, every one of which has its curvature continually varying, either increaſing or decreaſing, and every point having a degree of curvature peculiar to itſelf; and the radius of a circle which has the ſame curvature with the curve at any given point, is the radius of curvature at that point; which radius it is the buſineſs of this chapter to find.



74. Let AEe be any curve, concave toward its axis AD; draw an ordinate DE to the point E, where the curvature is required to be found; and ſuppoſe EC perpendicular to the curve, and equal to the radius of curvature ſought, or equal to the radius of a circle having the ſame curvature there, and with that radius deſcribe the ſaid equally-curved circle BEe;

BEe; laſtly, draw Ed parallel to AD, and de parallel and indefinitely near to DE; thereby making Ed the fluxion, or increment of the abſciſs AD, alſo de the fluxion of the ordinate DE, and Ee that of the curve AE. Then put $x =$ AD, $y =$ DE, $z =$ AE, and $r =$ CE the radius of curvature; then is E$d = \dot{x}$, $de = \dot{y}$, and E$e = \dot{z}$.

Now, by ſim. triangles, the three lines Ed, de, Ee,
or $\dot{x}$, $\dot{y}$, $\dot{z}$,
are reſpectively as the three - - GE, GC, CE;
therefore - - - - - - GC . $\dot{x} =$ GE . $\dot{y}$;
and the flux. of this eq. is GC . $\ddot{x}$ + G$\dot{\text{C}}$. $\dot{x}$ = GE . $\ddot{y}$ + G$\dot{\text{E}}$. $\dot{y}$,
or, becauſe G$\dot{\text{C}}$ = —B$\dot{\text{G}}$, it is, GC . $\ddot{x}$ — B$\dot{\text{G}}$. $\dot{x}$ = GE . $\ddot{y}$ + G$\dot{\text{E}}$. $\dot{y}$.

But, ſince the two curves AE and BE have the ſame curvature at the point E, their abſciſſes and ordinates have the ſame fluxions at that point, that is, Ed or $\dot{x}$ is the fluxion both of AD and BG, and de or $\dot{y}$ is the fluxion both of DE and GE. In the equation above, therefore, ſubſtitute $\dot{x}$ for B$\dot{\text{G}}$, and $\dot{y}$ for G$\dot{\text{E}}$, and it becomes

$$\text{GC}\ddot{x} - \dot{x}\dot{x} = \text{GE}\ddot{y} + \dot{y}\dot{y},$$
$$\text{or} \quad \text{GC}\ddot{x} - \text{GE}\ddot{y} = \dot{x}^2 + \dot{y}^2 = \dot{z}^2.$$

Now, multiply the three terms of this equation reſpectively by theſe three quantities, $\frac{\dot{y}}{\text{GC}}$, $\frac{\dot{x}}{\text{GE}}$, $\frac{\dot{z}}{\text{CE}}$, which are equal, and it becomes - - $\dot{y}\ddot{x} - \dot{x}\ddot{y} = \frac{\dot{z}^3}{\text{CE}}$ or $\frac{\dot{z}^3}{r}$;

and hence is found $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}$, for the general value of the radius of curvature, for all curves whatever, in terms of the fluxions of the abſciſs and ordinate.

75. Farther, as in any caſe either x or y may be ſuppoſed to flow equably, that is, either $\dot{x}$ or $\dot{y}$ conſtant quantities, or $\ddot{x}$ or $\ddot{y}$ equal to nothing, it follows that, by this ſuppoſition, either of the terms in the denominator, of the value of r, may be made to vaniſh. Thus, when $\dot{x}$ is ſuppoſed conſtant, $\ddot{x}$ being then = 0, the value of r is barely - - - - - $\frac{\dot{z}^3}{-\dot{x}\ddot{y}}$; or r is $= \frac{\dot{z}^3}{\dot{y}\ddot{x}}$ when $\dot{y}$ is conſtant.

EXAMPLES.

EXAM. 1. To find the radius of curvature to any point of

of a parabola, whofe equation is $ax = y^2$, its vertex being A, and axis AD.

Now, the equation to the curve being $ax = y^2$; the fluxion of it is $a\dot{x} = 2y\dot{y}$; and the fluxion of this again is $a\ddot{x} = 2\dot{y}^2$, fuppofing $\dot{y}$ conftant; hence then r or

$$\frac{\dot{z}^3}{\dot{y}\ddot{x}} \text{ or } \frac{(\dot{x}^2+\dot{y}^2)^{\frac{3}{2}}}{\dot{y}\ddot{x}} \text{ is } = \frac{(a^2+4y^2)^{\frac{3}{2}}}{2a^2} \text{ or } \frac{(a+4x)^{\frac{3}{2}}}{2\sqrt{a}},$$

for the general value of the radius of curvature at any point E, the ordinate to which cuts off the abfcifs AD $= x$.

Hence, when the abfcifs x is nothing, the laft expreffion becomes barely $\frac{1}{2}a = r$, for the radius of curvature at the vertex of the parabola; that is, the diameter of the circle of curvature at the vertex of a parabola, is equal to a, the parameter of the axis.

EXAM. 2. To find the radius of curvature of an ellipfe, whofe equation is $a^2y^2 = c^2 \cdot \overline{ax - x^2}$.

$$\text{Anf. } r = \frac{(a^2c^2 + 4(a^2 - c^2) \times (ax - x^2))^{\frac{3}{2}}}{2a^4c}.$$

EXAM. 3. To find the radius of curvature of an hyperbola, whofe equation is $a^2y^2 = c^2 \cdot \overline{ax + x^2}$.

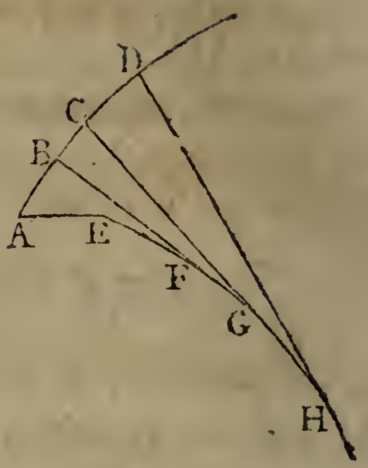
EXAM. 4. To find the radius of curvature of the cycloid. Anf. $r = 2\sqrt{aa - ax}$, where x is the abfcifs, and a the diameter of the generating circle.

OF INVOLUTE AND EVOLUTE CURVES.

76. AN Evolute is any curve fuppofed to be evolved or opened, by having a thread wrapped clofe about it, faftened at one end, and beginning to evolve or unwind the thread from the other end, keeping always tight ftretched the part which is evolved or wound off: then this end of the thread will defcribe another curve, called the Involute. Or, the fame involute is defcribed in the contrary way, by wrapping the thread about the curve of the evolute, keeping it at the fame time always ftretched.

77. Thus,

77. Thus, if EFGH be any curve, and AE be either a part of the curve, or a right line: then if a thread be fixed to the curve at H, and be wound or plyed close to the curve, &c, from H to A, keeping the thread always ſtretched tight; the other end of the thread will deſcribe a certain curve ABCD, called an Involute; the firſt curve EFGH being its evolute. Or, if the thread, fixed at H, be unwound from the curve, beginning at A, and keeping it always tight, it will deſcribe the ſame involute ABCD.



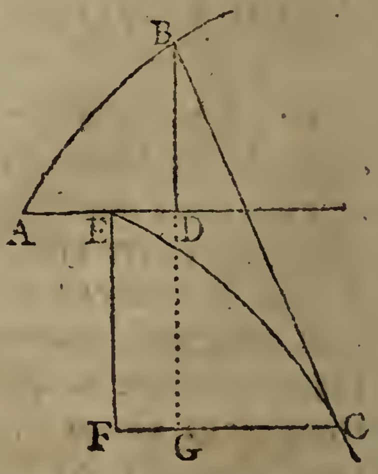
78. If AE, BF, CG, DH, &c, be any poſitions of the thread, in evolving or unwinding; it follows, that theſe parts of the thread are always the radii of curvature, at the corresponding points, A, B, C, D; and alſo equal to the correſponding lengths AE, AEF, AEFG, AEFGH, of the evolute; that is,

AE = AE is the radius of curvature to the point A,
BF = AF is the radius of curvature to the point B,
CG = AG is the radius of curvature to the point C,
DH = AH is the radius of curvature to the point D.

79. It alſo follows, from the premiſes, that any radius of curvature, BF, is perpendicular to the involute at the point B, and is a tangent to the evolute curve at the point F. Alſo, that the evolute is the locus of the centre of curvature of the involute curve.

80. Hence, and from art. 74, it will be eaſy to find one of theſe curves, when the other is given. To this purpoſe, put

x = AD, the abſciſs of the involute,
y = DB, an ordinate to the ſame,
z = AB, the involute curve,
r = BC, the radius of curvature,
v = EF, the abſciſs of the evolute EC,
u = FG, the ordinate of the ſame, and
a = AE, a certain given line.



Then,

Then, by the nature of the radius of curvature, it is $r = \frac{\dot{z}^3}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = BC = AE + EC$; alſo, by ſim triangles,

$$\dot{z} : \dot{x} :: r : GB = \frac{r\dot{x}}{\dot{z}} = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}};$$

$$\dot{z} : \dot{y} :: r : GC = \frac{r\dot{y}}{\dot{z}} = \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}}.$$

Hence $EF = GB - DB = \frac{\dot{x}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} - y = v$;

and $FC = AD - AE + GC = x - a + \frac{\dot{y}\dot{z}^2}{\dot{y}\ddot{x} - \dot{x}\ddot{y}} = u$;

which are the values of the abſciſs and ordinate of the evolute curve EC; from which, therefore, theſe may be found, when the involute is given.

On the contrary, if v and u, or the evolute, be given: then, putting the given curve $EC = s$; ſince $CB = AE + EC$, or $r = a + s$, this gives r the radius of curvature. Alſo, by ſimilar triangles, there ariſe theſe proportions, viz.

$$\dot{s} : \dot{v} :: r : \frac{r\dot{v}}{\dot{s}} = \frac{a+s}{\dot{s}}\dot{v} = GB,$$

and $\dot{s} : \dot{u} :: r : \frac{r\dot{u}}{\dot{s}} = \frac{a+s}{\dot{s}}\dot{u} = GC$;

theref. $AD = AE + FC - GC = a + u - \frac{a+s}{\dot{s}}\dot{u} = x$,

and $DB = GB - GD = GB - EF = \frac{a+s}{\dot{s}}\dot{v} - v = y$;

which are the abſciſs and ordinate of the involute curve, and which may therefore be found, when the evolute is given. Where it may be noted, that $\dot{s}^2 = \dot{v}^2 + \dot{u}^2$, and $\dot{z}^2 = \dot{x}^2 + \dot{y}^2$. Alſo, either of the quantities x, y, may be ſuppoſed to flow equably, in which caſe the reſpective ſecond fluxion, $\ddot{x}$ or $\ddot{y}$, will be nothing, and the correſponding term in the denominator $\dot{y}\ddot{x} - \dot{x}\ddot{y}$ will vaniſh, leaving only the other term in it; which will have the effect of rendering the whole operation ſimpler.

81. EXAMPLES.

EXAM. 1. To determine the nature of the curve by whoſe evolution the common parabola AB is deſcribed.

 Here

Here the equation of the given involute AB, is $cx = y^2$, where c is the parameter of the axis AD. Hence then $y = \sqrt{cx}$, and $\dot{y} = \frac{1}{2}\dot{x}\sqrt{\frac{c}{x}}$, alfo $\ddot{y} = \frac{-\dot{x}^2}{4x}\sqrt{\frac{c}{x}}$ by making $\dot{x}$ conftant. Confequently the general values of v and u, or of the abfcifs and ordinate, EF and FC, above given, become, in that cafe,

$$EF = v = \frac{\dot{z}^2}{-\ddot{y}} - y = \frac{\dot{x}^2 + \dot{y}^2}{-\ddot{y}} - y = 4x\sqrt{\frac{x}{c}}; \text{ and}$$

$$FC = u = x - a + \frac{\dot{y}\dot{z}^2}{-\dot{x}\ddot{y}} = 3x + \tfrac{1}{2}c - a.$$

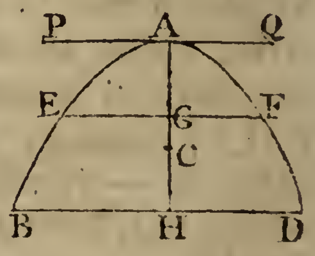
But the value of the quantity a or AE, by exam. 1 to art. 75, was found to be $\frac{1}{2}c$; confequently the laft quantity, FC or u, is barely $= 3x$.

Hence then, comparing the values of v and u, there is found $3v\sqrt{c} = 4u\sqrt{x}$, or $27cv^2 = 16u^3$; which is the equation between the abfcifs and ordinate of the evolute curve EC, fhewing it to be the femicubical parabola.

Exam. 2. To determine the evolute of the common cycloid. Anf. another cycloid, equal to the former.

To find the CENTRE of GRAVITY.

82. By referring to prop. 41, &c, in Mechanics, it is feen what are the principles and nature of the Centre of Gravity in any figure, and how it is generally expreffed. It there appears, that if PAQ be a line, or plane, drawn through any point, as fuppofe the vertex of any body, or figure, ABD, and if s denote any fection EF of the figure, d = AG, its diftance below PQ, and b = the whole body or figure ABD; then the diftance AC, of the centre of gravity below PQ, is univerfally denoted by $\frac{\text{fum of all the } ds}{b}$; whether ABD be a line, or a plane furface, or a curve fuperficies, or a folid.



But

But the ſum of all the ds, is the ſame as the fluent of $d\dot{b}$, and b is the ſame as the fluent of $\dot{b}$; therefore the general expreſſion for the diſtance of the centre of gravity, is AC $= \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent } x\dot{b}}{b}$; putting $x = d$ the variable diſtance AG. Which will divide into the following four caſes.

83. CASE 1. When AE is ſome line, as a curve ſuppoſe. In this caſe, $\dot{b}$ is $= \dot{z}$ or $\sqrt{\dot{x}^2 + \dot{y}^2}$, the fluxion of the curve; and $b = z$: theref. AC $= \frac{\text{fluent of } x\dot{z}}{z} = \frac{\text{fluent of } x\sqrt{\dot{x}^2 + \dot{y}^2}}{z}$ is the diſtance of the centre of gravity in a curve.

84. CASE 2. When the figure ABD is a plane; then $\dot{b} = y\dot{x}$; therefore the general expreſſion becomes AC $= \frac{\text{fluent of } yx\dot{x}}{\text{fluent of } y\dot{x}}$ for the diſtance of the centre of gravity in a plane.

85. CASE 3. When the figure is the ſuperficies of a body generated by the rotation of a line AEB, about the axis AH.

Then, putting $c = 3{\cdot}14159$ &c, $2cy$ will denote the circumference of the generating circle, and $2cy\dot{z}$ the fluxion of the ſurface; therefore AC $= \frac{\text{fluent of } 2cyx\dot{z}}{\text{fluent of } 2cy\dot{z}} = \frac{\text{fluent of } yx\dot{z}}{\text{fluent of } y\dot{z}}$ will be the diſtance of the centre of gravity for a ſurface generated by the rotation of a curve line z.

86. CASE 4. When the figure is a ſolid generated by the rotation of a plane ABH, about the axis AH.

Then, putting $c = 3{\cdot}14159$ &c, it is $cy^2 =$ the area of the circle whoſe radius is y, and $cy^2\dot{x} = \dot{b}$, the fluxion of the ſolid; therefore

$$AC = \frac{\text{fluent of } x\dot{b}}{\text{fluent of } \dot{b}} = \frac{\text{fluent of } cy^2x\dot{x}}{\text{fluent of } cy^2\dot{x}} = \frac{\text{fluent of } y^2x\dot{x}}{\text{fluent of } y^2\dot{x}}$$

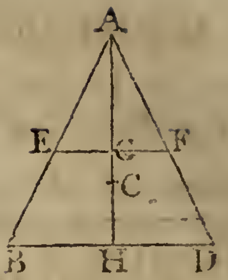
is the diſtance of the centre of gravity below the vertex in a ſolid.

87. EXAMPLES.

EXAM. 1. Let the figure propoſed be the iſoſceles triangle ABD.

It is evident that the centre of gravity C, will be ſome-

 where



where in the perpendicular AH. Now, if a denote AH, $c = $ BD, $x = $ AG, and $y = $ EF any line parallel to the base BD: then as $a : c :: x : y = \frac{cx}{a}$; therefore, by the 2d Case, AC $= \frac{\text{fluent } yx\dot{x}}{\text{fluent } y\dot{x}} = \frac{\text{fluent } x^2\dot{x}}{\text{fluent } x\dot{x}} = \frac{\frac{1}{3}x^3}{\frac{1}{2}x^2}$ $= \frac{2}{3}x = \frac{2}{3}$AH, when x becomes $=$ AH: consequently CH $= \frac{1}{3}$AH.

In like manner, the centre of gravity of any other plane triangle, will be found to be at $\frac{1}{3}$ of the altitude of the triangle; the same as it was found in prop. 42, Mechanics.

Exam. 2. In a parabola; the distance from the vertex is $\frac{3}{5}x$, or $\frac{3}{5}$ of the axis.

Exam. 3. In a circular arc; the distance from the centre of the circle, is $\frac{ac}{r}$; where a denotes the arc, c its chord, and r the radius.

Exam. 4. In a circular sector; the distance from the centre of the circle, is $\frac{cr}{a}$: where a, c, r, are the same as in exam. 3.

Exam. 5. In a circular segment; the distance from the centre of the circle is $\frac{c^3}{12a}$; where c is the chord, and a the area, of the segment.

Exam. 6. In a cone, or any other pyramid; the distance from the vertex is $\frac{3}{4}x$, or $\frac{3}{4}$ of the altitude.

Exam. 7. In the semisphere, or semispheroid; the distance from the centre is $\frac{3}{8}r$, or $\frac{3}{8}$ of the radius.

Exam. 8. In the parabolic conoid; the distance from the base is $\frac{1}{3}x$, or $\frac{1}{3}$ of the axis.

Exam. 9. In the segment of a sphere, or of a spheroid; the distance from the base is $\frac{2a - x}{6a - 4x}x$; where x is the height of the segment, and a the whole axis, or diameter of the sphere.

Exam. 10. In the hyperbolic conoid; the distance from the base is $\frac{2a + x}{6a + 4x}x$; where x is the height of the conoid, and a the whole axis or diameter.

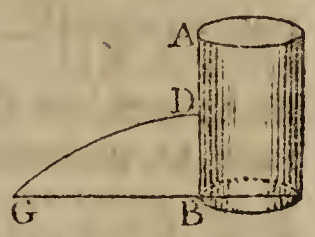
PRAC-

PRACTICAL QUESTIONS.

QUESTION I.

A LARGE veſſel, of 10 feet, or any other given depth, and of any ſhape, being kept conſtantly full of water, by means of a ſupplying cock at the top; it is propoſed to aſſign the place where a ſmall hole muſt be made in the ſide of it, ſo that the water may ſpout through it to the greateſt diſtance on the plane of the baſe.

Let AB denote the height or ſide of the veſſel; D the required hole in the ſide, from whence the water ſpouts, in the parabolic curve DG, to the greateſt diſtance BG, on the horizontal plane.

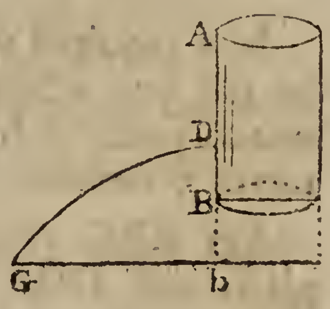


By the Scholium to prop. 61, Hydraulics, the diſtance BG is always equal to $2\sqrt{AD \cdot DB}$, which is equal to $2\sqrt{x(a-x)}$ or $2\sqrt{ax - x^2}$, if a be put to denote the whole height AB of the veſſel, and $x =$ AD the depth of the hole. Hence $2\sqrt{ax - x^2}$, or $ax - x^2$, muſt be a maximum. In fluxions, $a\dot{x} - 2x\dot{x} = 0$, or $a - 2x = 0$, and $2x = a$, or $x = \frac{1}{2}a$. So that the hole D muſt be in the middle between the top and bottom; the ſame as before found at the end of the ſcholium above quoted.

QUESTION II.

If the ſame veſſel, as in Queſt. 1, ſtand on high, with its bottom a given height above a horizontal plane below; it is propoſed to determine where the ſmall hole muſt be made, ſo as to ſpout fartheſt on the ſaid plane.

Let the annexed figure repreſent the veſſel as before, and bG the greateſt diſtance ſpouted by the fluid DG, on the plane bG.



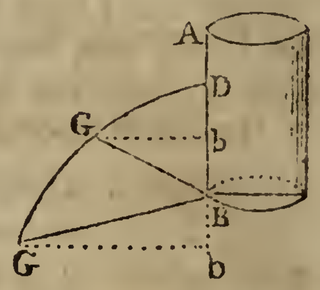
Here, as before, $bG = 2\sqrt{AD \cdot Db}$ $= 2\sqrt{x(c-x)} = 2\sqrt{cx - x^2}$, by putting $Ab = c$, and AD $= x$. So that $2\sqrt{cx - x^2}$ or $cx - x^2$ muſt be a maximum. And hence, like as in the former queſtion, $x = \frac{1}{2}c = \frac{1}{2}$AB. So that the hole D muſt be made in the

middle

middle between the top of the veſſel, and the given plane, that the water may ſpout fartheſt.

QUESTION III.

But if the ſame veſſel as before, ſtand on the top of an inclined plane, making a given angle, as ſuppoſe of 30 degrees, with the horizon; it is propoſed to determine the place of the ſmall hole, ſo as the water may ſpout the fartheſt on the ſaid inclined plane.



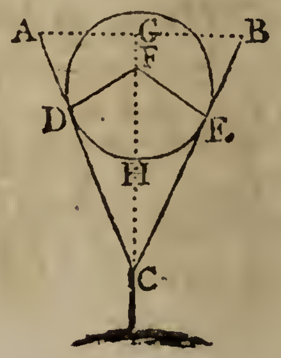
Here again (D being the place of the hole, and BG the given inclined plane), $bG = 2\sqrt{AD \cdot Db} = 2\sqrt{x(a - x \pm z)}$, putting $z = Bb$, and, as before, $a = AB$, and $x = AD$. Then bG muſt ſtill be a maximum, as alſo Bb, being in a given ratio to the maximum BG, on account of the given angle B. Therefore $ax - x^2 \pm xz$, as well as z, is a maximum. Hence, by art. 54 of the Fluxions, $a\dot{x} - 2x\dot{x} \pm z\dot{x} = 0$, or $a - 2x \pm z = 0$; conſeq. $\pm z = 2x - a$; and hence $bG = 2\sqrt{x(a - x \pm z)}$ becomes barely $2x$. But, as the given angle GBb is $= 30°$, the ſine of which is $\frac{1}{2}$; therefore $BG = 2Bb$ or $2z$, and $bG^2 = BG^2 - Bb^2 = 3z^2 = 3(2x - a)^2$, or $bG = \pm(2x - a)\sqrt{3}$.

Putting now theſe two values of bG equal to each other, gives the equation $2x = \pm(2x - a)\sqrt{3}$, from which is found $x = \dfrac{\frac{1}{2}a\sqrt{3}}{\sqrt{3} \mp 1} = \dfrac{3 \mp \sqrt{3}}{4}a$, the value of AD required.

Note. In the Select Exerciſes, page 165, this anſwer is brought out $\dfrac{6 + \sqrt{6}}{10}a$, by taking the velocity as the root of *half* the altitude.

QUESTION IV.

It is required to determine the ſize of a ball, which, being let fall into a conical glaſs full of water, ſhall expel the moſt water poſſible from the glaſs; its depth being 6, and diameter 5 inches.



Let ABC repreſent the cone of the glaſs, and DHE the ball, touching the ſides in the points D and E, the centre of the ball being at ſome point F in the axis GC of the cone.

Put

Put AG $=$ GB $= 2\frac{1}{2} = a$,
GC $= 6 = b$.
AC $= \sqrt{AG^2 + GC^2} = 6\frac{1}{2} = c$,
FD $=$ FE $=$ FH $= x$ the radius of the ball.

The two triangles ACG and DCF are equiangular; theref. AG : AC :: DF : FC, that is, $a : c :: x : \frac{cx}{a} =$ FC; hence GF $=$ GC $-$ FC $= b - \frac{cx}{a}$, and GH $=$ GF $+$ FH $= b + x - \frac{cx}{a}$, the height of the segment immersed in the water. Then (by rule 1 for the spherical segment, page 48), the content of the said immersed segment will be $(6\text{DF} - 2\text{GH}) \times \text{GH}^2 \times \cdot 5236 = (2x - b + \frac{cx}{a}) \times (x + b - \frac{cx}{a})^2 \times 1 \cdot 0472$, which must be a maximum by the question; the fluxion of this made $= 0$, and divided by $2\dot{x}$ and the common factors, gives $\frac{2a + c}{a} \times (b - \frac{c - a}{a} x) - (\frac{2a + c}{a} x - b) \times \frac{c - a}{a} \times 2 = 0$; this reduced gives $x = \frac{abc}{(c - a) \times (c + 2a)} = 2\frac{4}{9}\frac{1}{2}$, the radius of the ball. Consequently its diameter is $4\frac{1}{4}\frac{1}{4}$ inches, as required.

PRACTICAL EXERCISES, CONCERNING FORCES; WITH THE RELATION BETWEEN THEM AND THE TIME, VELOCITY, AND SPACE DESCRIBED.

BEFORE entering on the following problems, it will be convenient here, to lay down a synopsis of the theorems which express the several relations between any forces, and their corresponding times, velocities, and spaces described; which are all comprehended in the following 12 theorems, as collected from the principles in the foregoing parts of this work.

Let f, F be any two constant accelerative forces, acting on any body, during the respective times t, T, at the end of which are generated the velocities v, V, and described the spaces s, S. Then, because the spaces are as the times and velocities conjointly, and the velocities as the forces and times; we shall have,

I. In

I. *In Conſtant Forces.*

$$1.\quad \frac{s}{S} = \frac{tv}{TV} = \frac{t^2 f}{T^2 F} = \frac{v^2 F}{V^2 f}.$$

$$2.\quad \frac{v}{V} = \frac{ft}{FT} = \frac{sT}{St} = \sqrt{\frac{fs}{FS}}.$$

$$3.\quad \frac{t}{T} = \frac{Fv}{fV} = \frac{sV}{Sv} = \sqrt{\frac{Fs}{fS}}.$$

$$4.\quad \frac{f}{F} = \frac{Tv}{tV} = \frac{T^2 s}{t^2 S} = \frac{v^2 S}{V^2 s}.$$

And if one of the forces, as F, be the force of gravity at the ſurface of the earth, and be called 1, and its time T be $= 1''$; then it is known by experiment that the correſponding ſpace S is $= 16\frac{1}{12}$ feet, and conſequently its velocity V $= 2S = 32\frac{1}{6}$, which call $2g$, namely, $g = 16\frac{1}{12}$ feet, or 193 inches. Then the above four theorems, in this caſe, become as here below:

$$5.\quad s = \tfrac{1}{2}tv = gft^2 = \frac{v^2}{4gf}.$$

$$6.\quad v = \frac{2s}{t} = 2gft = \sqrt{4gfs}.$$

$$7.\quad t = \frac{2s}{v} = \frac{v}{2gf} = \sqrt{\frac{s}{gf}}.$$

$$8.\quad f = \frac{v}{2gt} = \frac{s}{gt^2} = \frac{v^2}{4gs}.$$

And from theſe are deduced the following four theorems, for variable forces, viz.

II. *In Variable Forces.*

$$9.\quad \dot{s} = v\dot{t} = \frac{v\dot{v}}{2gf}.$$

$$10.\quad \dot{v} = 2gf\dot{t} = \frac{2gf\dot{s}}{v}.$$

$$11.\quad \dot{t} = \frac{\dot{s}}{v} = \frac{\dot{v}}{2gf}.$$

$$12.\quad f = \frac{v\dot{v}}{2g\dot{s}} = \frac{\dot{v}}{2g\dot{t}}.$$

In

In these last four theorems, the force f, though variable, is supposed to be constant for the indefinitely small time $\dot{t}$, and they are to be used in all cases of variable forces, as the former ones in constant forces; namely, from the circumstances of the problem under consideration, an expression is deduced for the value of the force f, which being substituted in one of these theorems, that may be proper to the case in hand; the equation thence resulting will determine the corresponding values of the other quantities, required in the problem.

When a motive force happens to be concerned in the question, it may be proper to observe, that the motive force m, of a body, is equal to fq, the product of the accelerative force, and the quantity of matter in it q; and the relation between these three quantities being universally expressed by this equation $m = qf$, it follows that, by means of it, any one of the three may be expelled out of the calculation, or else brought into it.

Also, the momentum, or quantity of motion in a moving body, is qv, the product of the velocity and matter.

It is also to be observed, that the theorems equally hold good for the destruction of motion and velocity, by means of retarding forces, as for the generation of the same, by means of accelerating forces.

And to the following problems, which are all resolved by the application of these theorems, it has been thought proper to subjoin their solutions, for the better information and convenience of the student.

PROBLEM I.

To determine the time and velocity of a body descending, by the force of gravity, down an inclined plane; the length of the plane being 20 *feet, and its height* 1 *foot.*

Here, by Mechanics, the force of gravity being to the force down the plane, as the length of the plane is to its height, therefore as $20 : 1 :: 1$ (the force of gravity) $: \frac{1}{20} = f$, the force on the plane.

Therefore, by theor. 6, v or $\sqrt{4gfs}$ is $\sqrt{4 \times 16\frac{1}{12} \times \frac{1}{20} \times 20} = \sqrt{4 \times 16\frac{1}{12}} = 2 \times 4\frac{1}{96} = 8\frac{1}{48}$ feet, the last velocity per second. And,

By theor. 7, t or $\sqrt{\frac{s}{gf}}$, is $\sqrt{\frac{20}{16\frac{1}{12} \times \frac{1}{20}}} = \sqrt{\frac{400}{16\frac{1}{12}}} = \frac{20}{4\frac{1}{96}}$

$4\frac{1}{96} = 4\frac{76}{77}$ seconds, the time of descending.

PRO-

PROBLEM II.

If a cannon ball be fired with a velocity of 1000 *feet per second, up a smooth inclined plane, which rises* 1 *foot in* 20 : *it is proposed to assign the length which it will ascend up the plane, before it stops and begins to return down again, and the time of its ascent.*

Here $f = \frac{1}{20}$ as before.

Then, by theor. 5, $s = \frac{v^2}{4gf} = \frac{1000^2}{4 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{60000000}{193}$ $= 310880\frac{160}{193}$ feet, or nearly 59 miles, the distance moved.

And, by theor. 7, $t = \frac{v}{2gf} = \frac{1000}{2 \times 16\frac{1}{12} \times \frac{1}{20}} = \frac{120000}{193} =$ $621''\frac{147}{193} = 10'\ 21''\frac{147}{193}$, the time of ascent.

PROBLEM III.

If a ball be projected up a smooth inclined plane, which rises 1 *foot in* 10, *and ascend* 100 *feet before it stop: required the time of ascent, and the velocity of projection.*

First, by theor. 6, $v = \sqrt{4gfs} = \sqrt{4 \times 16\frac{1}{12} \times \frac{1}{10} \times 100} = 8\frac{1}{48}\sqrt{10} = 25{\cdot}36408$ feet per second, the velocity.

And, by theor. 7, $t = \sqrt{\frac{s}{gf}} = \sqrt{\frac{100}{16\frac{1}{12} \times \frac{1}{10}}} = \frac{10}{4\frac{1}{96}}\sqrt{10} = \frac{192}{77}\sqrt{10} = 7{\cdot}88516$ seconds, the time in motion.

PROBLEM IV.

If a ball be observed to ascend up a smooth inclined plane, 100 *feet in ten seconds, before it stop, to return back again: required the velocity of projection, and the angle of the plane's inclination.*

First, by theor. 6, $v = \frac{2s}{t} = \frac{200}{10} = 20$ feet per second, the velocity.

And, by theor. 8, $f = \frac{s}{gt^2} = \frac{100}{16\frac{1}{12} \times 100} = \frac{12}{193}$. That is, the length of the plane is to its height, as 193 to 12.

Therefore, 193 : 12 :: 100 : 6·2176 the height of the plane, or the sine of elevation to radius 100, which answers to 3° 34′, the angle of elevation of the plane.

PRO-

PROBLEM V.

By a mean of ſeveral experiments, I have found, that a caſt iron ball, of 2 inches diameter, impinging perpendicularly on the face or end of a block of elm wood, or in the direction of the fibres, with a velocity of 1500 feet per ſecond, penetrated 13 inches deep into its ſubſtance. It is propoſed then to determine the time of the penetration, and the reſiſting force of the wood, as compared to the force of gravity, ſuppoſing that force to be a conſtant quantity.

First, by theor. 7, $t = \frac{2s}{v} = \frac{2 \times 13}{1500 \times 12} = \frac{1}{692}$ part of a ſecond, the time in penetrating.

And, by theor. 8, $f = \frac{v^2}{4gs} = \frac{1500^2}{4 \times 16\frac{1}{12} \times \frac{13}{12}} = \frac{81000000}{13 \times 193} = 32284$. That is, the reſiſting force of the wood, is to the force of gravity, as 32284 to 1.

But this number will be different, according to the diameter of the ball, and its denſity or ſpecific gravity. For, ſince f is as $\frac{v^2}{s}$ by theor. 4, the denſity and ſize of the ball remaining the ſame; if the denſity, or ſpecific gravity, n, vary, and all the reſt be conſtant, it is evident that f will be as n; and therefore f as $\frac{nv^2}{s}$ when the ſize of the ball only is conſtant. But when only the diameter d varies, all the reſt being conſtant, the force of the blow will vary as d^3, or as the magnitude of the ball; and the reſiſting ſurface, or force of reſiſtance, varies as d^2; therefore f is as $\frac{d^3}{d^2}$, or as d only when all the reſt are conſtant. Conſequently f is as $\frac{dnv^2}{s}$ when they are all variable.

And ſo $\frac{f}{F} = \frac{dnv^2S}{DNV^2s}$, and $\frac{s}{S} = \frac{dnv^2F}{DNV^2f}$; where f denotes the ſtrength or firmneſs of the ſubſtance penetrated, and is here ſuppoſed to be the ſame, for all balls and velocities, in the ſame ſubſtance, which it is either accurately or nearly ſo. See page 264, &c, of my Tracts, vol. i.

Hence, taking the numbers in the problem, it is - - -
$f = \frac{dnv^2}{s} = \frac{\frac{2}{12} \times 7\frac{1}{5} \times 1500^2}{\frac{13}{12}} = \frac{44 \times 1500^2}{39} = 2538462$ the value of f for elm wood. Where the ſpecific gravity of the

the ball is taken $7\frac{1}{3}$, which is a little lefs than that of folid caft iron, as it ought, on account of the air bubble which is found in all caft balls.

PROBLEM VI.

To find how far a 24lb *ball of caft iron will penetrate into a block of found elm, when fired with a velocity of* 1600 *feet per fecond.*

Here, becaufe the fubftance is the fame as in the laft problem, both of the balls and wood, $N = n$, and $F = f$; therefore $\frac{S}{s} = \frac{DV^2}{dv^2}$, or $S = \frac{DV^2 s}{dv^2} = \frac{5\cdot55 \times 1600^2 \times 13}{2 \times 1500^2} =$ $41\frac{2}{45}$ inches nearly, the penetration required.

PROBLEM VII.

It was found by Mr. Robins (vol. i. p. 273, *of his works*), *that an* 18 *pounder ball, fired with a velocity of* 1200 *feet per fecond, penetrated* 34 *inches into found dry oak. It is required to afcertain the comparative ftrength or firmnefs of oak and elm.*

The diameter of an 18lb ball is 5·04 inches $= D$. Then, by the numbers given in this problem for oak, and in prob. 5 for elm, we have

$$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 34}{5\cdot04 \times 1200^2 \times 13} = \frac{100 \times 17}{5\cdot04 \times 16 \times 13} = \frac{1700}{1048}$$

or $= \frac{8}{5}$ nearly.

From which it would feem, that elm timber refifts more than oak, in the ratio of about 8 to 5; which is not probable, as oak is a much firmer and harder wood. But it is to be fufpected that Mr. Robins's great penetration was owing to the fplitting of his timber in fome degree.

PROBLEM VIII.

A 24 *pounder ball being fired into a bank of firm earth, with a velocity of* 1300 *feet per fecond, penetrated* 15 *feet: It is required then to afcertain the comparative refiftances of elm and earth.*

Comparing the numbers here with thofe in prob. 5, it is $\frac{f}{F}$

$\frac{f}{F} = \frac{dv^2 S}{DV^2 s} = \frac{2 \times 1500^2 \times 15 \times 12}{5\cdot55 \times 1300^2 \times 13} = \frac{15^2 \times 24}{13^3 \times 0\cdot37} = \frac{1800}{271} = \frac{20}{3}$ nearly $= 6\frac{2}{3}$ nearly. That is, elm timber reſiſts about $6\frac{2}{3}$ times more than earth.

PROBLEM IX.

To determine how far a leaden bullet, of $\frac{3}{4}$ of an inch diameter, will penetrate dry elm; ſuppoſing it fired with a velocity of 1700 feet per ſecond, and that the lead does not change its figure by the ſtroke againſt the wood.

HERE $D = \frac{3}{4}$, $N = 11\frac{1}{3}$, $n = 7\frac{1}{3}$. Then, by the numbers and theorem in prob. 5, it is $S = - - - - - - - - \frac{DNV^2 s}{dnv^2} = \frac{\frac{3}{4} \times 11\frac{1}{3} \times 1700^2 \times 13}{2 \times 7\frac{1}{3} \times 1500^2} = \frac{17^3 \times 13}{200 \times 33} = \frac{63869}{6600} = 9\frac{2}{3}$ inches nearly, the depth of penetration.

But as Mr. Robins found this penetration, by experiment, to be only 5 inches; it follows, either that his timber muſt have reſiſted about twice as much; or elſe, which is much more probable, that the defect in his penetration aroſe from the change of figure in the leaden ball, from the blow againſt the wood.

PROBLEM X.

A one pound ball, projected with a velocity of 1500 feet per ſecond, having been found to penetrate 13 inches deep into dry elm: It is required to aſcertain the time of paſſing through every ſingle inch of the 13, and the velocity loſt at each of them; ſuppoſing the reſiſtance of the wood conſtant or uniform.

THE velocity v being 1500 feet, or $1500 \times 12 = 18000$ inches, and velocities and times being as the roots of the ſpaces, in conſtant retarding forces, as well as in accelerating ones, and t being $= \frac{2s}{v} = \frac{26}{12 \times 1500} = \frac{13}{9000} = \frac{1}{692}$ part of a ſecond, the whole time of paſſing through the 13 inches; therefore as

$\sqrt{13}$

$$\sqrt{13} : \sqrt{13} - \sqrt{12} :: v :$$

veloc. loſt		Time in the	
$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}} v = 58{\cdot}9 :: t :$	$\frac{\sqrt{13}-\sqrt{12}}{\sqrt{13}} t = {\cdot}00005$	1ſt
$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}} v = 61{\cdot}4 :: t :$	$\frac{\sqrt{12}-\sqrt{11}}{\sqrt{13}} t = {\cdot}00006$	2d
$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}} v = 64{\cdot}2$ &c	$\frac{\sqrt{11}-\sqrt{10}}{\sqrt{13}} t = {\cdot}00006$	3d
$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}} v = 67{\cdot}5$	$\frac{\sqrt{10}-\sqrt{9}}{\sqrt{13}} t = {\cdot}00007$	4th
$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}} v = 71{\cdot}4$	$\frac{\sqrt{9}-\sqrt{8}}{\sqrt{13}} t = {\cdot}00007$	5th
$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}} v = 76{\cdot}0$	$\frac{\sqrt{8}-\sqrt{7}}{\sqrt{13}} t = {\cdot}00007$	6th
$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}} v = 81{\cdot}7$	$\frac{\sqrt{7}-\sqrt{6}}{\sqrt{13}} t = {\cdot}00008$	7th
$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}} v = 88{\cdot}8$	$\frac{\sqrt{6}-\sqrt{5}}{\sqrt{13}} t = {\cdot}00008$	8th
$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}} v = 98{\cdot}2$	$\frac{\sqrt{5}-\sqrt{4}}{\sqrt{13}} t = {\cdot}00009$	9th
$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}} v = 111{\cdot}4$	$\frac{\sqrt{4}-\sqrt{3}}{\sqrt{13}} t = {\cdot}00011$	10th
$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}} v = 132{\cdot}2$	$\frac{\sqrt{3}-\sqrt{2}}{\sqrt{13}} t = {\cdot}00013$	11th
$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}} v = 172{\cdot}3$	$\frac{\sqrt{2}-\sqrt{1}}{\sqrt{13}} t = {\cdot}00017$	12th
$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}} v = 416{\cdot}0$	$\frac{\sqrt{1}-\sqrt{0}}{\sqrt{13}} t = {\cdot}00040$	13th
Sum $1500{\cdot}0$	Sum $\frac{1}{692}$ or ${\cdot}00144$	inch.

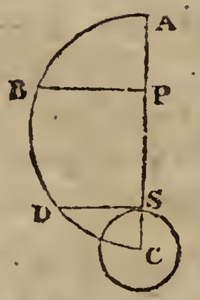
Hence, as the motion loſt at the beginning is very ſmall; and conſequently the motion communicated to any body, as an inch plank, in paſſing through it, is very ſmall alſo; we can conceive how ſuch a plank may be ſhot through, without overſetting it.

PRO-

PROBLEM XI.

The force of attraction, above the earth, being inverſely as the ſquare of the diſtance from the centre; it is propoſed to determine the time, velocity, and other circumſtances, attending a heavy body falling from any given height; the deſcent at the earth's ſurface being $16\frac{1}{12}$ *feet, or* 193 *inches, in the firſt ſecond of time.*

Put



$r =$ CS the radius of the earth,
$a =$ CA the diſt. fallen from,
$x =$ CP any variable diſtance,
$v =$ the velocity at P,
$t =$ time of falling there, and
$g = 16\frac{1}{12}$, half the veloc. or force at S,
$f =$ the force at the point P.

Then we have the three following equations, viz.

$x^2 : r^2 :: 1 : f = \frac{r^2}{x^2}$ the force at P, when the force of gravity is conſidered as 1;
$\dot{t}v = -\dot{x}$, becauſe x decreaſes; and
$v\dot{v} = -2gf\dot{x} = -\frac{2gr^2\dot{x}}{x^2}$.

The fluents of the laſt equation give $v^2 = \frac{4gr^2}{x}$.

But when $x = a$, the velocity $v = 0$; therefore, by correction, $v^2 = \frac{4gr^2}{x} - \frac{4gr^2}{a} = 4gr^2 \times \frac{a - x}{ax}$; or $v = \sqrt{\frac{4gr^2}{a} \times \frac{a - x}{x}}$, a general expreſſion for the velocity at any point P.

When $x = r$, this gives $v = \sqrt{4gr \times \frac{a - r}{a}}$ for the greateſt velocity, or the velocity when the body ſtrikes the earth.

When a is very great in reſpect of r, the laſt velocity becomes $(1 - \frac{r}{2a}) \times \sqrt{4gr}$ very nearly, or nearly $\sqrt{4gr}$ only, which is accurately the greateſt velocity by falling from an infinite height. And this, when $r = 3965$ miles, is 6·9506 miles per ſecond. Alſo, the velocity acquired in falling from the

the diſtance of the ſun, or 12000 diameters of the earth, is 6·9505 miles per ſecond. And the velocity acquired in falling from the diſtance of the moon, or 30 diameters, is 6·8927 miles per ſecond.

Again, to find the time; ſince $\dot{t}v = -\dot{x}$, therefore $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{a}{4gr^2}} \times \frac{-x\dot{x}}{\sqrt{ax - xx}}$; the correct fluent of which gives $t = \sqrt{\frac{a}{4gr^2}} \times (\sqrt{ax - xx}$ + arc to diameter a and verſ. $a - x)$; or the time of falling to any point P $= \frac{1}{2r}\sqrt{\frac{a}{g}} \times (\text{AB} + \text{BP})$. And when $x = r$, this becomes $t = \frac{1}{2}\sqrt{\frac{a}{g}} \times \frac{\text{AD} + \text{DS}}{\text{SC}}$ for the whole time of falling to the ſurface at S; which is evidently infinite when a or AC is infinite, although the velocity is then only the finite quantity $\sqrt{4gr}$.

When the height above the earth's ſurface is given $= g$; becauſe r is then nearly $= a$, and AD nearly $=$ DS, the time t for the diſtance g will be nearly

$\sqrt{\frac{1}{4gr^2}} \times 2\text{DS} = \sqrt{\frac{1}{4gr}} \times \sqrt{4gr} = 1''$, as it ought to be.

If a body at the diſtance of the moon at A fall to the earth's ſurface at S. Then $r = 3965$ miles, $a = 60r$, and $t = 416806'' = 4$ da. 19h. 46′ 46″, which is the time of falling from the moon to the earth.

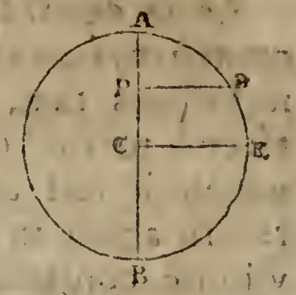
When the attracting body is conſidered as a point C; the whole time of deſcending to C will be - - - - -

$\frac{1}{2r}\sqrt{\frac{a}{g}} \times \text{ABDC} = \frac{\cdot 7854a}{r}\sqrt{\frac{a}{g}}$.

PROBLEM XII.

The force of attraction below the earth's ſurface being directly as the diſtance from the centre: it is propoſed to determine the circumſtances of velocity, time, and ſpace fallen by a heavy body from the ſurface, through a perforation made ſtraight to the centre of the earth: abſtracting from the effect of the earth's rotation, and ſuppoſing it to be a homogeneous ſphere of 3965 miles radius.

Put



Put $r = AC$ the radius of the earth,
$x = CP$ the dist. from the centre,
$v =$ the velocity at P,
$t =$ the time there,
$g = 16\frac{1}{12}$, half the force at A,
$f =$ the force at P.

Then CA : CP :: 1 : f; and the three equations are $rf = x$, and $v\dot{v} = -2gf\dot{x}$, and $\dot{t}v = -\dot{x}$.

Hence $f = \frac{x}{r}$, and $v\dot{v} = \frac{-2gx\dot{x}}{r}$; the correct fluent of which gives $v = \sqrt{2g \times \frac{r^2 - x^2}{r}} = PD\sqrt{\frac{2g}{r}} = PD\sqrt{\frac{2g}{CE}}$, the velocity at the point P; where PD and CE are perpendicular to CA. So that the velocity at any point P, is as the perpendicular or sine PD at that point.

When the body arrives at C, then $v = \sqrt{2gr} = \sqrt{2g \, . \, AC} = 25950$ feet or 4·9148 miles per second, which is the greatest velocity, or that at the centre C.

Again, for the time; $\dot{t} = \frac{-\dot{x}}{v} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{r^2 - x^2}}$; and the fluents give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{r} = \sqrt{\frac{1}{2gr}} \times$ arc AD. So that the time of descent to any point P, is as the corresponding arc AD.

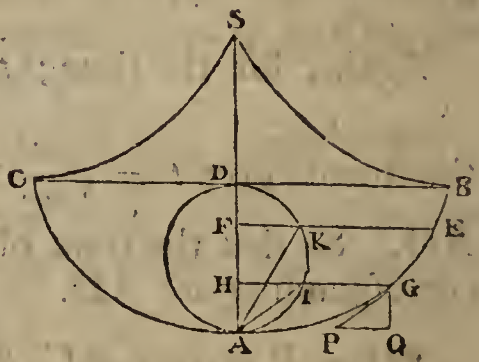
When P arrives at C, the above becomes $t = \sqrt{\frac{1}{2gr}} \times$ quadrant $AE = \frac{AE}{AC}\sqrt{\frac{r}{2g}} = 1{\cdot}5708\sqrt{\frac{r}{2g}} = 1267\frac{1}{4}$ seconds $= 21'\,7''\frac{1}{4}$, for the time of falling to the centre C.

The time of falling to the centre, is the same quantity $1{\cdot}5708\sqrt{\frac{r}{2g}}$, from whatever point in the radius AC the body begins to move. For, let n be any given distance from C at which the motion commences: then, by correction, $v = \sqrt{\frac{2g}{r}(n^2 - x^2)}$; and hence $\dot{t} = \sqrt{\frac{r}{2g}} \times \frac{-\dot{x}}{\sqrt{n^2 - x^2}}$, the fluents of which give $t = \sqrt{\frac{r}{2g}} \times$ arc to cosine $\frac{x}{n}$; which, when $x = 0$, gives $t = \sqrt{\frac{r}{2g}} \times$ quadrant $= 1{\cdot}5708\sqrt{\frac{r}{2g}}$, for the time of descent to the centre C, the same as before.

As an equal force, acting in contrary directions, generates or destroys an equal quantity of motion, in the same time; it follows that, after passing the centre, the body will just ascend to the opposite surface at B, in the same time in which it fell to the centre from A. Then from B it will return again in the same manner, through C to A; and so vibrate continually between A and B, the velocity being always equal at equal distances from C on both sides; and the whole time of a double oscillation, or of passing from A and arriving at A again, will be quadruple the time of passing over the radius AC, or $= 2 \times 3.1416\sqrt{\frac{r}{2g}} =$ 1h. 24′ 29″.

PROBLEM XIII.

To find the Time of a Pendulum vibrating in the Arc of a Cycloid.



Let
S be the point of suspension;
SA, the length of pendulum;
CAB, the whole cycloidal arc;
AIKD, the generating circle, to which FKE, HIG are perpendiculars.
SC, SB two other equal semicycloids, upon which the thread wrapping, the end A is made to describe the cycloid BAC.

Now, by the nature of the cycloid, AD = DS; and SA = 2AD = SC = SB = SA = AB. Also, if at any point G be drawn the tangent GP; also GQ parallel and PQ perpendicular to AD. Then PG is parallel to the chord AI by the nature of the curve. And, by the nature of forces, the force of gravity : force in direction GP : : GP : GQ : : AI : AH : : AD : AI; in like manner, the force of gravity : force in the curve at E : : AD : AK; that is, the accelerative force in the curve, is every where as the corresponding chord AI or AK of the circle, or as the arc AG or AE of the cycloid, since AG is always = 2AI, by the nature of the curve. So that the process and conclusions, for the velocity and time of describing any arc in this case, will be the very same as in the last problem, the nature of the forces being the same, viz. as the distance to be passed over to the lowest point A.

From

From whence it follows, that the time of a ſemi-vibration, in all arcs, AG, AE, &c, is the ſame conſtant quantity $1\cdot5708\sqrt{\frac{r}{2g}} = 1\cdot5708\sqrt{\frac{AS}{2g}} = 1\cdot5708\sqrt{\frac{l}{2g}}$; and the time of a whole vibration from B to C, or from C to B, is $3\cdot1416\sqrt{\frac{l}{2g}}$; where $l = AS = AB$ is the length of the pendulum, $g = 16\frac{1}{12}$ feet or 193 inches, and 3·1416 the circumference of a circle whoſe diameter is 1.

Since the time of a body's falling by gravity through $\frac{1}{2}l$, or half the length of the pendulum, by the nature of deſcents, is $\sqrt{\frac{l}{2g}}$, which being in proportion to $3\cdot1416\sqrt{\frac{l}{2g}}$, as 1 is to 3·1416; therefore the diameter of a circle is to its circumference, as the time of falling through half the length of a pendulum, is to the time of one vibration.

If the time of the whole vibration be 1 ſecond, this equation ariſes, viz. $1'' = 3\cdot1416\sqrt{\frac{l}{2g}}$; hence $l = \frac{2g}{3\cdot1416^2} = \frac{g}{4\cdot9348}$, and $g = 3\cdot1416^2 \times \frac{1}{2}l = 4\cdot9348l$. So that if one of theſe, g or l, be given by experiment, theſe equations will give the other. When g, for inſtance, is ſuppoſed to be given $= 16\frac{1}{12}$ feet, or 193 inches; then is $l = \frac{g}{4\cdot9348} = 39\cdot11$, the length of a pendulum to vibrate ſeconds. Or if $l = 39\frac{1}{8}$, the length of the ſeconds pendulum for the latitude of London, by experiment; then is $g = 4\cdot9348l = 193\cdot07$ inches $= 16\frac{107}{1200}$ feet, or nearly $16\frac{1}{12}$ feet, for the ſpace deſcended by gravity in the firſt ſecond of time in the latitude of London; alſo agreeing with experiment.

Hence the times of vibration of pendulums, are as the ſquare roots of their lengths; and the number of vibrations made in a given time, is reciprocally as the ſquare roots of the lengths. And hence alſo, the length of a pendulum vibrating n times in a minute, or $60''$, is $l = 39\frac{1}{8} \times \frac{60^2}{n^2} = \frac{140850}{nn}$.

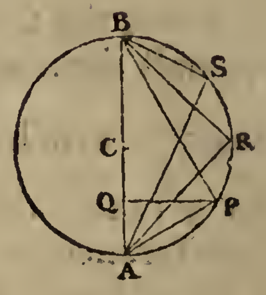
When a pendulum vibrates in a circular arc; as the length of the ſtring is conſtantly the ſame, the time of vibration will be longer than in a cycloid; but the two times will approach nearer together as the circular arc is ſmaller; ſo that

 when

when it is very ſmall, the times of vibration will be nearly equal. And hence it happens that $39\frac{1}{8}$ inches is the length of a pendulum vibrating ſeconds, in the very ſmall arc of a circle.

PROBLEM XIV.

To determine the Time of a Body deſcending down the Chord of a Circle.



LET C be the centre; AB the vertical diameter; AP any chord, down which a body is to deſcend from P to A; and PQ perpendicular to AB.

Now, as the natural force of gravity in the vertical direction BA, is to the force urging the body down the plane PA, as the length of the plane AP, is to its height AQ; therefore the velocity in PA and QA, will be equal at all equal perpendicular diſtances below PQ; and conſequently the time in PA : time in QA :: PA : QA :: BA : PA; but time in BA : time in QA :: $\sqrt{BA} : \sqrt{QA}$ :: BA : PA; hence, as three of the terms in each proportion are the ſame, the fourth terms muſt be equal, namely the time in BA = the time PA.

And, in like manner, the time in BP = the time in BA. So that, in general, the times of deſcending down all the chords BA, BP, BR, BS, &c, or PA, RA, SA, &c, are all equal, and each equal to the time of falling freely through the diameter; as before found at art. 131, Mechanics. Which time is $\sqrt{\frac{2r}{g}}$, where $g = 16\frac{1}{12}$ feet, and $r =$ the radius AC; for $\sqrt{g} : \sqrt{2r} :: 1'' : \sqrt{\frac{2r}{g}}$.

PROBLEM XV.

To determine the Time of filling the Ditches of a Work with Water, at the Top, by a Sluice of 2 Feet ſquare; the Head of Water above the Sluice being 10 Feet, and the Dimenſions of the Ditch being 20 Feet wide at Bottom, 22 at Top, 9 deep, and 1000 Feet long.

THE capacity of the ditch is 189000 cubic feet.

But $\sqrt{g} : \sqrt{10} :: 2g : 2\sqrt{10g}$ the velocity of the water through the ſluice, the area of which is 4 ſquare feet; therefore

therefore $8\sqrt{10g}$ is the quantity per fecond running through it; and confequently $8\sqrt{10g} : 189000 :: 1'' : \frac{23625}{\sqrt{10g}} = 1863''$ or $31'\ 3''$ nearly, which is the time of filling the ditch.

PROBLEM XVI.

To determine the Time of emptying a Veffel of Water by a Sluice in the Bottom of it, or in the Side near the Bottom; the Height of the Aperture being very fmall in refpect of the Altitude of the Fluid.

PUT $a =$ the area of the aperture or fluice;
$2g = 32\frac{1}{6}$ feet, the force of gravity;
$d =$ the whole depth of water;
$x =$ the variable altitude of the furface above the aperture;
$A =$ the area of the furface of the water.

Then $\sqrt{g} : \sqrt{x} :: 2g : 2\sqrt{gx}$ the velocity with which the fluid will iffue at the fluice; and hence $A : a :: 2\sqrt{gx} : \frac{2a\sqrt{gx}}{A}$ the velocity with which the furface of the water will defcend at the altitude x, or the fpace it would defcend in 1 fecond with the velocity there. Now, in defcending the fpace $\dot{x}$, the velocity may be confidered as uniform; and uniform defcents are as their times; therefore $\frac{2a\sqrt{gx}}{A} : \dot{x} :: 1'' : \frac{A\dot{x}}{2a\sqrt{gx}}$ the time of defcending $\dot{x}$ fpace, or the fluxion of the time of exhaufting. That is, $\dot{t} = \frac{-A\dot{x}}{2a\sqrt{gx}}$; which is made negative, becaufe x is a decreafing quantity, or its fluxion negative.

Then, when the nature or figure of the veffel is given, the area A will be given in terms of x; which value of A being fubftituted into this fluxion of the time, the fluent of the refult will be the time of exhaufting fought.

So if, for example, the veffel be any prifm, or every where of the fame breadth; then A is a conftant quantity, and therefore the fluent is $-\frac{A}{a}\sqrt{\frac{x}{g}}$. But when $x = d$, this becomes $-\frac{A}{a}\sqrt{\frac{d}{g}}$, and fhould be 0; therefore the correct fluent is $t = \frac{A}{a} \times \frac{\sqrt{d} - \sqrt{x}}{\sqrt{g}}$ for the time of the furface de-

fcending

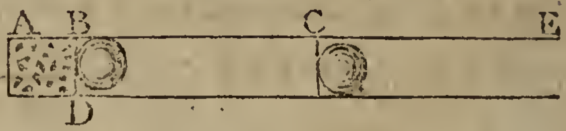
scending till the depth of the water be x. And when $x=0$, the whole time of exhausting is barely $\frac{A}{a}\sqrt{\frac{d}{g}}$.

Hence, if A be 10000 square feet, $a=1$ square foot, and $d=10$ feet; the time is $7885\frac{1}{5}$ seconds, or 2h. 11′ 25″$\frac{1}{5}$.

Again, if the vessel be a ditch, or canal, of 20 feet broad at the bottom, 22 at the top, 9 deep, and 1000 feet long; then is $90 : 90+x :: 20 : \frac{90+x}{9}\times 2$ the breadth of the surface of the water when its depth in the canal is x; and therefore $A=\frac{90+x}{9}\times 2000$ is the surface at that time. Consequently $\dot{t}$ or $\frac{-A\dot{x}}{2a\sqrt{gx}} = 1100\times\frac{90+x}{9}\times\frac{-\dot{x}}{a\sqrt{gx}}$ is the fluxion of the time; the correct fluent of which, when $x=0$, is $1000\times\frac{180+\frac{2}{3}d}{9a}\times\sqrt{\frac{d}{g}}=\frac{1000\times186\times3}{9\times4\frac{1}{96}}=$ 15459″$\frac{2}{3}$ nearly, or 4h. 17′ 39″$\frac{2}{3}$, being the whole time of exhausting by a sluice of 1 foot square.

PROBLEM XVII.

To determine the Velocity with which a Ball is discharged from a Given Piece of Ordnance, with a Given Charge of Gunpowder.



Let the annexed figure represent the bore of the gun; AD being the part filled with gunpowder. And put

$a=$ AB, the part at first filled with powder;
$b=$ AE, the whole length of the gun bore;
$c=\cdot7854$, the area of a circle whose diameter is 1;
$d=$ BD, the diameter of the bore;
$e=$ the specific gravity of the ball, or weight of 1 cubic foot;
$g=16\frac{1}{12}$ feet, descended by a body in 1 second;
$m=$ 230 ounces, the pressure of the atmosphere on a sq. inch;
n to 1 the ratio of the first force of the fired powder, to the pressure of the atmosphere;
$w=$ the weight of the ball. Also, let
$x=$ AC, be any variable distance of the ball from A, in moving along the gun barrel.

First,

First, cd^2 is $\doteq$ the area of the circle BD of the bore; theref. mcd^2 is the pressure of the atmosphere on BD; conseq. $mncd^2$ is the first force of the powder on BD.

But the force of the inflamed powder is proportional to its density, and the density is inversely as the space it fills; therefore the force of the powder on the ball at B, is to the force on the same at C, as AC is to AB; that is,

$x : a :: mncd^2 : \frac{mnacd^2}{x} = F$, the motive force at C;

conseq. $\frac{F}{w} = \frac{mnacd^2}{wx} = f$, the accelerating force there.

Hence, theor. 10 of forces gives $v\dot{v} = 2gf\dot{x} = \frac{2gmnacd^2}{w} \times \frac{\dot{x}}{x}$;

the fluent of which is $v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. of x.

But when $v = 0$, $x = a$; therefore, by correction,

$v^2 = \frac{4gmnacd^2}{w} \times$ hyp. log. $\frac{x}{a}$ is the correct fluent;

conseq. $v = \sqrt{(\frac{4gmnacd^2}{w} \times \text{hyp. log. } \frac{x}{a})}$ is the vel. of ball at C,

and $v = \sqrt{(\frac{4gmnbcd^2}{w} \times \text{hyp. log. } \frac{b}{a}}$ the velocity with which the ball issues from the muzzle at E; where b denotes the length of the cylinder filled with powder, and a the length to the hinder part of the ball, which will be more than b when the powder does not touch the ball.

Or, by substituting the numbers for g and m, and changing the hyperbolic logarithms for the common ones, then

$v = \sqrt{(\frac{2230nbd^2}{w} \times \text{com. log. } \frac{b}{a})}$ the veloc. at E, in feet.

But, the content of the ball being $\frac{2}{3}cd^3$, its weight is

$w = \frac{\frac{2}{3}cd^3e}{12^3} = \frac{ced^3}{2592} = \frac{ed^3}{3300}$; which being substituted for w, in the value of v, it becomes

$v = 2713\sqrt{(\frac{nb}{de} \times \text{com. log. } \frac{b}{a})}$, the velocity at E.

When the ball is of cast iron; taking $e = 7333$, the rule becomes

$v = 100\sqrt{(\frac{nb}{10d} \times \text{log. } \frac{b}{a})}$ for the veloc. of the cast-iron ball.

Or, when the ball is of lead; then

$v = 80\frac{3}{5}\sqrt{(\frac{nb}{10d} \times \text{log. } \frac{b}{a})}$ for the veloc. of the leaden ball.

Corol.

Corol. From the general expreſſion for the velocity v, above given, may be derived what muſt be the length of the charge of powder a, in the gun barrel, ſo as to produce the greateſt poſſible velocity in the ball; namely, by making the value of v a maximum, or, by ſquaring and omitting the conſtant quantities, the expreſſion $a \times$ hyp. log. of $\frac{b}{a}$ a maximum, or its fluxion equal to nothing; that is, $\dot{a} \times$ hyp. log. $\frac{b}{a} - \dot{a} = 0$, or hyp. log. of $\frac{b}{a} = 1$; hence $\frac{b}{a} = 2{\cdot}71828$, the number whoſe hyp. log. is 1. So that $a : b :: 1 : 2{\cdot}71828$, or as $7 : 19$ very nearly; that is, the length of the charge, to produce the greateſt velocity, is the $\frac{7}{19}$th part of the length of the bore.

By actual experiment it is found, that the charge for the greateſt velocity, ſomewhat exceeds that which is here computed from theory; as may be ſeen by turning to page 269 of my volume of Tracts, where the correſponding parts are found to be, for four different lengths of gun, thus, $\frac{3}{10}$, $\frac{3}{12}$, $\frac{3}{16}$, $\frac{3}{20}$; the parts here varying, as the gun is longer, which allows time for the greater quantity of powder to be fired, before the ball is out of the bore.

SCHOLIUM.

In the calculation of the foregoing problem, the value of the conſtant quantity n remains to be determined. It denotes the firſt ſtrength or force of the fired gunpowder, juſt before the ball is moved out of its place. This value is aſſumed, by Mr. Robins, equal to 1000, that is, 1000 times the preſſure of the atmoſphere, on any equal ſpaces.

But the value of the quantity n, may be derived much more accurately, from the experiments related in my Tracts, by comparing the velocities there found by experiment, with the rule for the value of v, or the velocity, as above computed by theory, viz. - - - - - - - - - - - - - -

$$v = 100\sqrt{\left(\frac{na}{10d} \times \text{log. of } \frac{b}{a}\right)}, \text{ or } = 100\sqrt{\left(\frac{nb}{10d} \times \text{log. of } \frac{b}{a}\right)}.$$

Now, ſuppoſing that v is a given quantity, as well as all the other quantities, excepting only the number n, by reducing this equation, the value of the letter n is found to be as follows, viz. - - - - - - - - - - - - - - - -

$$n = \frac{dvv}{1000a} \div \text{com. log. of } \frac{b}{a}, \text{ or } = \frac{dvv}{1000b} \div \text{log. of } \frac{b}{a},$$

when b is different from a.

Now,

Now, to apply this to the experiments. By page 257 of the Tracts, the velocity of the ball, of 1·96 inches diameter, with 4 ounces of powder, in the gun No. 1, was 1100 feet per ſecond; and, by page 109, the length of the gun was 28·2; alſo, by page 237, the length of the charge was 3·12 inches: ſo that the values of the quantities in the rule, are thus: $a = 3·12$; $b = 28·2$; $d = 1·96$; and $v = 1100$: then, by ſubſtituting theſe values inſtead of the letters, in the theorem $n = \frac{dvv}{1000a} \div$ com. log. of $\frac{b}{a}$, it comes out $n = 795$, when b is conſidered as the ſame as a. And ſo on, for the other experiments there treated of.

It is here to be noted however, that there is a circumſtance in the experiments delivered in the Tracts, juſt mentioned, which will alter the value of the letter a in this theorem, which is this, viz. that a denotes the diſtance of the ſhot from the bottom of the bore; and the length of the charge of powder alone ought to be the ſame thing; but in the experiments that length included, beſide the length of real powder, the ſubſtance of the thin flannel bag in which it was always contained, of which the neck at leaſt extended a conſiderable length, being the part where the open end was wrapped and tied cloſe round with a thread. This circumſtance cauſes the value of n, as found by the theorem above, to come out leſs than it ought to be, for it ſhews the ſtrength of the inflamed powder when juſt fired, and the flame fills the whole ſpace a before occupied both by the real powder and the bag, whereas it ought to ſhew the firſt ſtrength of the flame when it is ſuppoſed to be contained in the ſpace only occupied by the powder alone, without the bag. The formula will therefore bring out the value of n too little, in proportion as the real ſpace filled by the powder is leſs than the ſpace filled both by the powder and its bag. In the ſame proportion therefore muſt we increaſe the formula, that is, in the proportion of h, the length of real powder, to a the length of powder and bag together. When the theorem is ſo corrected, it becomes $\frac{dvv}{1000h} \div$ com. log. of $\frac{b}{a}$.

Now, by pa. 237 of the Tracts, there are given both the lengths of all the charges, or values of a, including the bag, and alſo the length of the neck of the bag, which is 0·58 of an inch, which therefore muſt be ſubtracted from all the values of a, to give the correſponding values of h. This in the example above reduces 3·12 to 2·54.

Hence,

Hence, by increasing the above result 795, in proportion of 2·54 to 3·12, it becomes 977.

But it will be best to arrange the results in a table, with the several dimensions, from which they are computed, as here below.

Table of Velocities of Balls, and First Force of Powder, &c.

Gun.		Charge.			Velocity, or value of v.	First force, or value of n.
No.	Length, or value of b.	Weight in ounces.	Length, or value. of a.	Length, or value. of h.		
	inches.					
1	28·2	4	3·12	2·54	1100	977
		8	5·66	5·08	1430	1131
		16	10·74	10·16	1430	941
2	38·1	4	3·12	2·54	1180	989
		8	5·66	5·08	1580	1163
		16	10·74	10·16	1660	967
3	57·37	4	3·12	2·54	1300	1031
		8	5·66	5·08	1790	1229
		16	10·74	10·16	2000	1060
4	79·9	4	3·12	2·54	1370	1028
		8	5·66	5·08	1940	1263
		16	10·74	10·16	2200	1071

where the numbers in the column of velocities, 1420 and 2200, are a little increased, as, from a view of the table of experiments, they evidently required to be. Also the value of the letter is constantly 1·36 inch.

Hence it appears, that the value of the letter n, used in the theorem, though not greatly different from the number 1000, used by Mr. Robins, is rather various, both for the different lengths of the gun, and for the different charges with the same gun.

But this diversity in the value of the quantity n, or the first force of the inflamed gunpowder, is probably owing to the omission of a material datum in the calculation of the last

laſt problem, namely, the weight of the charge of powder, which has not at all been brought into the computation. For it is manifeſt, that the elaſtic fluid has not only the ball to move and impel before it, but its own weight of matter alſo. The computation may therefore be renewed, in the enſuing problem, to take that datum into the account.

PROBLEM XVIII.

To determine the ſame as in the laſt Problem; taking both the Weight of Powder and the Ball into the Calculation.

Beside the notation uſed in the laſt problem, let $2p$ denote the weight of the powder in the charge.

Now, becauſe the inflamed powder occupies all the part of the gun bore which is behind the ball, its centre of gravity, or the middle part of the ſame, will move with only half the velocity that the ball moves with; and this will require the ſame force, as half the weight of the powder, moved with the ſame velocity as the ball. Therefore, in the concluſion derived in the laſt problem, we are now, inſtead of w, to ſubſtitute the quantity $p + w$; and when that is done, the laſt velocity will come out, $v = \sqrt{(\frac{2230nad^2}{p+w} \times \text{com. log. } \frac{b}{a})}$.

Now, to find the value of $p + w$ in terms of ad, the dimenſions of the ball and powder; it appears, in the firſt place, from the calculations in the laſt ſcholium, that about 5 inches in length of a gun, of near 2 inches diameter, contains juſt 8 ounces of powder; and as the contents of cylinders, or the weight of powder they will contain, are in the compound ratio of their length and ſquare of the diameter; therefore as $5 \times 2^2 : ad^2 :: 8 : \frac{2ad^2}{5}$, which will be the weight of powder contained in the bore, or charge, whoſe length is a, and diameter d; that is, $2p = \frac{2ad^2}{5}$, or $p = \frac{ad^2}{5}$, the value of the quantity p, in terms of the dimenſions of the charge.

But the value of the quantity w, or weight of the ball, as found in the laſt problem, was $w = \frac{ed^3}{3300}$. Conſeq. $p + w = \frac{ad^2}{5} + \frac{ed^3}{3300} = \frac{660ad^2 + ed^3}{3300}$.

Then, ſubſtituting this value of $p + w$, in the expreſſion above found for the velocity, gives

$v =$

$v = 2713\sqrt{\frac{na}{660a + de}} \times \log. \frac{b}{a}$, for the laſt veloc. of ball.

And, when the ball is of caſt iron, uſing 7333 for n, the ſame theorem becomes

$v = \sqrt{\frac{1000na}{d + \cdot 09a}} \times \log. \frac{b}{a}$, or $= \sqrt{\frac{1000nb}{d + \cdot 09b}} \times \log. \frac{b}{a}$, for the laſt velocity of the ball, when of caſt iron.

Corol. To determine the maximum velocity, from the foregoing expreſſion, by ſquaring, and expelling the conſtant quantities, there is obtained at length $\frac{a}{d + \cdot 09a} \times$ hyp. log. $\frac{b}{a}$ = a maximum. Then, by making the fluxion of this quantity = 0, there is produced hyp. log. $\frac{b}{a} = 1 + \cdot 09a$. From which a may eaſily be found, in any particular caſe, by the method of trial-and-error, and a table of hyperbolic logarithms.

Thus, taking the firſt caſe in the corollary to the laſt problem, in which $b = 28\cdot 2$, and $d = 2$ nearly; then, by double poſition, is ſoon found $a = 7\cdot 4$; which is $\frac{3}{11}$ of b, being very near $\frac{3}{10}b$, as found by experiment.

Again, taking the ſecond caſe there mentioned, in which $b = 38\cdot 1$, and $d = 2$, as before. Then is eaſily found $a = 9\cdot 3$; which is $\frac{1}{4}$ or $\frac{3}{12}$ of b, the very ſame as found by experiment.

Thirdly, taking the third caſe, in which $b = 57\cdot 37$, and $d = 2$. Then a comes out 12; which is between $\frac{1}{14}$ and $\frac{1}{15}$ of b; which was found $\frac{1}{16}b$ by experiment.

Laſtly, taking the fourth caſe, in which $b = 79\cdot 9$, and $d = 2$. Then a comes out 15, which is $\frac{3}{16}$ of b, but was found near $\frac{3}{20}$ of b by experiment.

The near agreement of theſe calculations, from theory, with the experiments, is a confirmation of the truth, both of the one and the other. And this gives a reaſonable ground to hope, that the value of the quantity n, or the firſt force of the fired gunpowder, when calculated from the laſt theorem for the velocity, may come nearly the ſame, when fired in the ſame quantity, by uſing different experiments. Which may now be tried in the following ſcholium.

SCHOLIUM.

Proceeding here as in the ſcholium to the laſt problem, to find the value of the letter n, in terms of v and the known quan-

quantities, from the theorem for the velocity laſt found, viz.

$$v = \sqrt{\frac{1000na}{d + \cdot 09a}} \times \log. \frac{b}{a}, \text{ or } = \sqrt{\frac{1000nb}{d + \cdot 09b}} \times \log. \frac{b}{a},$$

by reduction it comes out

$$n = (\frac{d + \cdot 09a)v^2}{1000a} \div \log. \frac{b}{a}, \text{ or } \frac{(d + \cdot 09b)v^2}{1000b} \div \log. \frac{b}{a}.$$

Then, calculating the value of n, by this theorem, from the ſame data as in the 12 experiments made uſe of in the ſcholium to the laſt problem, the values of n come out as they are here placed in the margin, which ſtand in the ſame order as thoſe in the laſt column of the table of data and reſults in the ſcholium before mentioned. Alſo, the correſponding values of the ſame force n, from the former problem and ſcholium, are ſet after theſe, that their differences may be ſeen at one view.

Values of n from this prob.	Values of n from the former prob.
1091	977
1396	1131
1381	941
1104	989
1435	1163
1419	967
1152	1031
1516	1229
1556	1060
1149	1028
1558	1263
1572	1071

From a view of theſe reſults, it clearly appears, that the theorems, for the values of v and n, found and employed in this problem, and the corollary and ſcholium to the ſame, muſt be very near the truth, as the values of n, or firſt force of the powder, come out very uniform, for all the guns, when the ſame quantity of powder is fired. To make this uniformity more obvious, they are here arranged, according to the like weight of powder in each column, and for each gun on the ſame line. From which it appears, that there is very little difference on account of the length of the gun; but a conſiderable one for the different quantities of powder, being, on an average, nearly, as follows, viz.

Length of gun.	Value of n, with powder.		
	4 oz.	8 oz.	16 oz.
inc.			
28·2	1091	1396	1381
38·1	1104	1435	1419
57·37	1152	1516	1556
79·9	1149	1558	1572

That

That the first force of fired gunpowder,

when 4 oz are fired, is about 1150 times	the strength of the atmosphere.
when 8 oz are fired, is about 1500 times	
when 16 oz are fired, is about 1550 times	

And this increase of strength is probably owing to the greater heat of the elastic fluid, arising from the greater quantity of powder that is fired in the gun.

On the MOTION of BODIES in FLUIDS.

PROBLEM XIX.

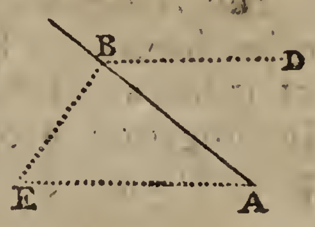
To determine the Force of Fluids in Motion; and the Circumstances attending Bodies moving in Fluids.

1. IT is evident that the resistance to a plane, moving perpendicularly through an infinite fluid, at rest, is equal to the pressure or force of the fluid on the plane at rest, and the fluid moving with the same velocity, and in the contrary direction, to that of the plane in the former case. But the force of the fluid in motion, must be equal to the weight or pressure which generates that motion; and which, it is known, is equal to the weight or pressure of a column of the fluid, whose base is equal to the plane, and its altitude equal to the height through which a body must fall, by the force of gravity, to acquire the velocity of the fluid: and that altitude is, for the sake of brevity, called the altitude due to the velocity. So that, if a denote the area of the plane, v the velocity, and n the specific gravity of the fluid; then, the altitude due to the velocity v being $\frac{v^2}{4g}$, the whole resistance, or motive force m, will be $a \times n \times \frac{v^2}{4g} = \frac{anv^2}{4g}$; g being $16\frac{1}{12}$ feet. And hence, *cæteris paribus*, the resistance is as the square of the velocity.

2. This ratio, of the square of the velocity, may be otherwise derived thus. The force of the fluid in motion, must be as the force of one particle multiplied by the number of them; but the force of a particle is as its velocity; and the number of them striking the plane in a given time, is also as the velocity; therefore the whole force is as $v \times v$ or v^2, that is, as the square of the velocity.

3. If the direction of motion, instead of being perpendicular to the plane, as above supposed, be inclined to it in any

any angle, the ſine of that angle being s, to the radius 1: then the reſiſtance to the plane, or the force of the fluid againſt the plane, in the direction of the motion, as aſſigned above, will be diminiſhed in the triplicate ratio of radius to the ſine of the angle of inclination, or in the ratio of 1 to s^3. For, AB being the direction of the plane, and BD that of the motion, making the angle ABD, whoſe ſine is s; the number of particles, or quantity of the fluid, ſtriking the plane, will be diminiſhed in the ratio of 1 to s, or of radius to the ſine of the angle B of inclination; and the force of each particle will alſo be diminiſhed in the ſame ratio of 1 to s: ſo that, on both theſe accounts, the whole reſiſtance will be diminiſhed in the ratio of 1 to s^2, or in the duplicate ratio of radius to the ſine of the ſaid angle. But again, it is to be conſidered that this whole reſiſtance is exerted in the direction BE perpendicular to the plane; and any force in the direction BE, is to its effect in the direction AE, parallel to BD, as AE to BE, that is as 1 to s. So that finally, on all theſe accounts, the reſiſtance in the direction of motion, is diminiſhed in the ratio of 1 to s^3, or in the triplicate ratio of radius to the ſine of inclination. Hence, comparing this with article 1, the whole reſiſtance, or the motive force on the plane, will be $m = \frac{anv^2s^3}{4g}$.



4. Alſo, if w denote the weight of the body, whoſe plane face a is reſiſted by the abſolute force m; then the retarding force f, or $\frac{m}{w}$, will be $\frac{anv^2s^3}{4gw}$.

5. And if the body be a cylinder, whoſe face or end is a, and diameter d, or radius r, moving in the direction of its axis; becauſe then $s = 1$, and $a = pr^2 = \frac{1}{4}pd^2$, where $p = 3{\cdot}1416$; the reſiſting force m will be - - - - - $\frac{npd^2v^2}{16g} = \frac{npr^2v^2}{4g}$, and the retarding force $f = \frac{npd^2v^2}{16gw} = \frac{npr^2v^2}{4gw}$.

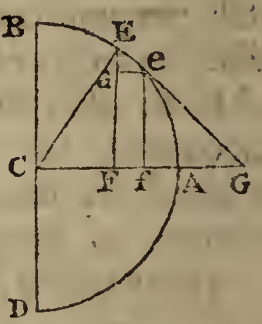
6. This is the value of the reſiſtance when the end of the cylinder is a plane perpendicular to its axis, or to the direction of motion. But were its face a conical ſurface, or an elliptic ſection, or any other figure every where equally inclined to the axis, the ſine of inclination being s: then, the number of

of particles of the fluid ſtriking the face being ſtill the ſame, but the force of each, oppoſed to the direction of motion, diminiſhed in the duplicate ratio of radius to the ſine of inclination, the reſiſting force m would be $\frac{npd^2v^2s^2}{16g} = \frac{npr^2v^2s^2}{4g}$.

But if the body were terminated by an end or face of any other form, as a ſpherical one, or ſuch like, where every part of it has a different inclination to the axis; then a farther inveſtigation becomes neceſſary, ſuch as in the following propoſition.

PROBLEM XX.

To determine the Reſiſtance of a Fluid to any Body, moving in it, of a Curved End; as a Sphere, or a Cylinder with a Hemiſpherical End, &c.



1. Let BEAD be a ſection through the axis CA of the ſolid, moving in the direction of that axis. To any point of the curve draw the tangent EG, meeting the axis produced in G: alſo, draw the perpendicular ordinates EF, ef indefinitely near to each other; and draw ae parallel to CG.

Putting CF $= x$, EF $= y$, BE $= z$, $s =$ ſine $\angle$G to radius 1, and $p = 3{\cdot}1416$: then $2py$ is the circumference whoſe radius is EF, or the circumference deſcribed by the point E, in revolving about the axis CA; and $2py \times$ Ee or $2py\dot{z}$ is the fluxion of the ſurface, or it is the ſurface deſcribed by Ee, in the ſaid revolution about CA, and which is the quantity repreſented by a in art. 3 of the laſt problem: hence $\frac{nv^2s^3}{4g} \times 2py\dot{z}$ or $\frac{pnv^2s^3}{2g} \times y\dot{z}$ is the reſiſtance on that ring, or the fluxion of the reſiſtance to the body, whatever the figure of it may be. And the fluent of which will be the reſiſtance required.

2. In the caſe of a ſpherical form; putting the radius CA or CB $= r$, we have $y = \sqrt{r^2 - x^2}$, $s = \frac{\text{EF}}{\text{EG}} = \frac{\text{CF}}{\text{CE}} = \frac{x}{r}$, and $y\dot{z}$ or EF $\times$ Ee $=$ CE $\times$ae $= r\dot{x}$; therefore the general fluxion $\frac{pnv^2}{2g} \times s^3y\dot{z}$ becomes $\frac{pnv^2}{2g} \times \frac{x^3}{r^3} \times r\dot{x} = \frac{pnv^2}{2gr^2} \times x^3\dot{x}$;

the

the fluent of which, or $\frac{pnv^2}{8gr^2}x^4$, is the resistance to the spherical surface generated by BE. And when x or CF is $= r$ or CA, it becomes $\frac{pnv^2r^2}{8g}$ for the resistance on the whole hemisphere; which is also equal to $\frac{pnv^2d^2}{32g}$, where $d = 2r$ the diameter.

3. But the perpendicular resistance to the circle of the same diameter d or BD, by art. 5 of the preceding problem, is $\frac{pnv^2d^2}{16g}$; which, being double the former, shews that the resistance to the sphere, is just equal to half the direct resistance to a great circle of it, or to a cylinder of the same diameter.

4. Since $\frac{1}{6}pd^3$ is the magnitude of the globe; if N denote its density or specific gravity, its weight w will be $= \frac{1}{6}pd^3N$, and therefore the retardive force f or $\frac{m}{w} = \frac{pnv^2d^2}{32g} \times \frac{6}{pNd^3} = \frac{3nv^2}{16gNd}$; which is also $= \frac{v^2}{4gs}$ by art. 8 of the general theorems in page 329; hence then $\frac{3n}{4Nd} = \frac{1}{s}$, and $s = \frac{N}{n} \times \frac{4}{3}d$; which is the space that would be described by the globe, while its whole motion is generated or destroyed by a constant force which is equal to the force of resistance, if no other force acted on the globe to continue its motion. And if the density of the fluid were equal to that of the globe, the resisting force is such, as, acting constantly on the globe without any other force, would generate or destroy its motion in describing the space $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter, by that accelerating or retarding force.

5. Hence the greatest velocity that a globe will acquire by descending in a fluid, by means of its relative weight in the fluid, will be found by making the resisting force equal to that weight. For, after the velocity is arrived at such a degree, that the resisting force is equal to the weight that urges it, it will increase no longer, and the globe will afterwards continue to descend with that velocity uniformly. Now, N and n being the separate specific gravities of the globe and fluid, N — n will be the relative gravity of the globe in the fluid, and therefore $w = \frac{1}{6}pd^3(N - n)$ is the

weight by which it is urged; alſo $m =$ - - - $\frac{pnv^2d^2}{32g}$ is the reſiſtance; conſequently $\frac{pnv^2d^2}{32g} = \frac{1}{6}pd^3(\mathrm{N}-n)$ when the velocity becomes uniform; from which equation is found $v = \sqrt{4g \cdot \frac{4}{3}d \cdot \frac{\mathrm{N}-n}{n}}$, for the ſaid uniform or greateſt velocity.

And, by comparing this form with that in art. 6 of the general theorems in page 329, it will appear that its greateſt velocity, is equal to the velocity generated by the accelerating force $\frac{\mathrm{N}-n}{n}$, in deſcribing the ſpace $\frac{4}{3}d$, or equal to the velocity generated by gravity in freely deſcribing the ſpace $\frac{\mathrm{N}-n}{n} \times \frac{4}{3}d$.—If $\mathrm{N} = 2n$, or the ſpecific gravity of the globe be double that of the fluid, then $\frac{\mathrm{N}-n}{n} = 1 =$ the natural force of gravity; and then the globe will attain its greateſt velocity in deſcribing $\frac{4}{3}d$, or $\frac{4}{3}$ of its diameter.—It is farther evident, that if the body be very ſmall, it will very ſoon acquire its greateſt velocity, whatever its denſity may be.

Exam. If a leaden ball, of 1 inch diameter, deſcend in water, and in air of the ſame denſity as at the earth's ſurface, the three ſpecific gravities being as $11\frac{1}{3}$, and 1, and $\frac{3}{2500}$. Then $v = \sqrt{4 \cdot 16\frac{1}{12} \cdot \frac{4}{36} \cdot 10\frac{1}{3}} = \frac{1}{9}\sqrt{31 \cdot 193} = 8\cdot5944$ feet, is the greateſt velocity per ſecond the ball can acquire by deſcending in water. And $v = \sqrt{4 \cdot \frac{193}{12} \cdot \frac{4}{36} \cdot \frac{34}{3} \cdot \frac{2500}{3}}$ nearly $= \frac{50}{9}\sqrt{\frac{34 \cdot 193}{3}} = 259\cdot82$ is the greateſt velocity it can acquire in air.

But if the globe were only $\frac{1}{100}$ of an inch diameter, the greateſt velocities it could acquire, would be only $\frac{1}{10}$ of theſe, namely $\frac{86}{100}$ of a foot in water, and 26 feet nearly in air. And if the ball were ſtill farther diminiſhed, the greateſt velocity would alſo be diminiſhed, and that in the ſubduplicate ratio of the diameter of the ball.

PRO-

PROBLEM XXI.

To determine the Relations of Velocity, Space, and Time, of a Ball moving in a Fluid, in which it is projected with a Given Velocity.

1. LET $a =$ the firſt velocity of projection, x the ſpace deſcribed in any time t, and v the velocity then. Now, by art. 4 of the laſt problem, the accelerative force $f = \frac{3nv^2}{16gNd}$, where N is the denſity of the ball, n that of the fluid, and d the diameter. Therefore the general equation $v\dot{v} = 2gf\dot{s}$ becomes $v\dot{v} =$ - - - - - - -
$-\frac{3nv^2}{8Nd}\dot{x}$; and hence $\frac{\dot{v}}{v} = \frac{-3n}{8Nd}\dot{x} = -b\dot{x}$, putting b for $\frac{3n}{8Nd}$.
The correct fluent of this, is log. a — log. v or log. $\frac{a}{v} = bx$. Or, putting $c = 2{\cdot}718281828$, the number whoſe hyp. log. is 1, then is $\frac{a}{v} = c^{bx}$, and the velocity $v = \frac{a}{c^{bx}} = ac^{-bx}$.

2. The velocity v at any time being the c^{-bx} part of the firſt velocity, therefore the velocity loſt in any time, will be the $1 - c^{-bx}$ part, or the $\frac{c^{bx} - 1}{c^{bx}}$ part of the firſt velocity.

EXAMPLES.

EXAM. 1. If a globe be projected, with any velocity, in a medium of the ſame denſity with itſelf, and it deſcribe a ſpace equal to $3d$ or 3 of its diameters. Then $x = 3d$, and $b = \frac{3n}{8Nd} = \frac{3}{8d}$; therefore $bx = \frac{9}{8}$, and $\frac{c^{bx} - 1}{c^{bx}} = \frac{2{\cdot}08}{3{\cdot}08}$ is the velocity loſt, or nearly $\frac{2}{3}$ of the projectile velocity.

EXAM. 2. If an iron ball of 2 inches diameter were projected with a velocity of 1200 feet per ſecond; to find the velocity loſt after moving through any ſpace, as ſuppoſe 500 feet of air: we ſhould have $d = \frac{2}{12} = \frac{1}{6}$, $a = 1200$,

$x = 500$, $N = 7\frac{1}{3}$, $n = \cdot 0012$; and therefore $bx =$ $\frac{3nx}{8Nd} = \frac{3 \,.\, 12 \,.\, 500 \,.\, 3 \,.\, 6}{8 \,.\, 22 \,.\, 10000} = \frac{81}{440}$, and $v = \frac{1200}{c^{\frac{81}{440}}} = 998$ feet per fecond: having loft 202 feet, or nearly $\frac{1}{6}$ of its firft velocity.

Exam. 3. If the earth revolved about the fun, in a medium as denfe as the atmofphere near the earth's furface; and it were required to find the quantity of motion loft in a year. Then, fince the earth's mean denfity is about $4\frac{1}{2}$, and its diftance from the fun 12000 of its diameters, we have $24000 \times 3\cdot1416 = 75398$ diameters $= x$, and $bx =$ - - - - - - $\frac{3 \,.\, 75398 \,.\, 12 \,.\, 2}{8 \,.\, 10000 \,.\, 9} = 7\cdot5398$; hence $\frac{c^{bx} - 1}{c^{bx}} = \frac{1880}{1881}$ parts are loft of the firft motion in the fpace of a year, and only the $\frac{1}{1881}$ part remains.

Exam. 4. If it be required to determine the diftance moved, x, when the globe has loft any part of its motion, as fuppofe $\frac{1}{2}$, and the denfity of the globe and fluid equal: The general equation gives $x = \frac{1}{b} \times \log. \frac{a}{v} = \frac{8d}{3} \times \log.$ of $2 = 1\cdot8483925d$. So that the globe lofes half its motion before it has defcribed twice its diameter.

3. To find the time t; we have $\dot{t} = \frac{\dot{s}}{v} = \frac{\dot{x}}{v} = \frac{c^{bx}\dot{x}}{a}$. Now, to find the fluent of this, put $z = c^{bx}$; then is $bx =$ log. z, and $b\dot{x} = \frac{\dot{z}}{z}$, or $\dot{x} = \frac{\dot{z}}{bz}$; confeq. $\dot{t}$ or $\frac{c^{bx}\dot{x}}{a} =$ $\frac{z\dot{x}}{a} = \frac{\dot{z}}{ab}$; and hence $t = \frac{z}{ab} = \frac{c^{bx}}{ab}$. But as t and x vanifh together, and when $x = 0$, the quantity $\frac{c^{bx}}{ab}$ is $= \frac{1}{ab}$; therefore, by correction, $t = \frac{c^{bx} - 1}{ab} = \frac{1}{bv} - \frac{1}{ba} = \frac{1}{b}(\frac{1}{v} - \frac{1}{a})$ the time fought; where $b = \frac{3n}{8Nd}$, and $v = \frac{a}{c^{bx}}$ the velocity.

Exam. If an iron ball of 2 inches diameter were projected in the air with a velocity of 1200 feet per fecond; and it were required to determine in what time it would pafs over

500 yards

500 yards or 1500 feet, and what would be its velocity at the end of that time: We ſhould have, as in exam. 2 above, $b = \frac{3 . 12 . 3 . 6}{8 . 22 . 10000} = \frac{1}{2716}$, and $bx = \frac{1500}{2716} = \frac{375}{679}$; hence $\frac{1}{b} = \frac{2716}{1}$, and $\frac{1}{a} = \frac{1}{1200}$, and $\frac{1}{v} = \frac{c^{bx}}{a} = \frac{1{\cdot}7372}{1200} = \frac{1}{690}$ nearly. Conſequently $v = 690$ is the velocity; and $t = \frac{1}{b}(\frac{1}{v} - \frac{1}{a}) = 2716 \times (\frac{1}{690} - \frac{1}{1200}) = 1\frac{31}{46}$ ſeconds, is the time required, or 1″ and $\frac{2}{3}$ nearly.

PROBLEM XXII.

To determine the Relations of Space, Time, and Velocity, when a Globe deſcends, by its own Weight, in an infinite Fluid.

THE foregoing notation remaining, viz. $d =$ diameter, N and n the denſity of the ball and fluid, and v, s, t, the velocity, ſpace, and time, in motion; we have $\frac{1}{6}pd^3 =$ the magnitude of the ball, and $\frac{1}{6}pd^3(\mathrm{N} - n) =$ its weight in the fluid, alſo $m = \frac{pnd^2v^2}{32g} =$ its reſiſtance from the fluid; conſequently $\frac{1}{6}pd^3(\mathrm{N} - n) - \frac{pnd^2v^2}{32g}$ is the motive force by which the ball is urged; which being divided by $\frac{1}{6}\mathrm{N}d^3$, the quantity of matter moved, gives $f = 1 - \frac{n}{\mathrm{N}} - \frac{3nv^2}{16g\mathrm{N}d}$ for the accelerative force.

2. Hence $v\dot{v} = 2gf\dot{s}$, and $\dot{s} = \frac{v\dot{v}}{2gf} = \frac{\mathrm{N}v\dot{v}}{2g(\mathrm{N} - n) - \frac{3n}{8d}v^2}$ $= \frac{1}{b} \times \frac{v\dot{v}}{a - v^2}$, putting $b = \frac{3n}{8\mathrm{N}d}$, and $\frac{1}{a} = \frac{3n}{2g . 8d(\mathrm{N} - n)}$, or $ab = 2g$ nearly; the fluent of which is $s = \frac{1}{2b} \times$ log. of $\frac{a}{a - v^2}$, an expreſſion for the ſpace s, in terms of the velocity v.

3. But now, to determine v in terms of s, put $c = 2{\cdot}718281828$; then, ſince the log. of $\frac{a}{a - v^2} = 2bs$, therefore

fore $\frac{a}{a-v^2} = c^{2bs}$, or $\frac{a-v^2}{a} = c^{-2bs}$; hence $v = \sqrt{a - ac^{-2bs}}$ is the velocity ſought.

4. The greateſt velocity is to be found, as in art. 5 of prob. 20, by making f or $1 - \frac{n}{N} - \frac{3nv^2}{16gNd} = 0$, which gives $v = \sqrt{2g \cdot 8d \cdot \frac{N-n}{3n}} = \sqrt{a}$. The ſame value is alſo obtained by making the fluxion of v^2, or of $a - ac^{-2bs}$, $= 0$. And the ſame value of v is obtained by making s infinite, for then $c^{-2bs} = 0$. But this velocity $\sqrt{a}$ cannot be attained in any finite time, and it only denotes the velocity to which the general value of v or $\sqrt{a - ac^{-2bs}}$ continually approaches. It is evident, however, that it will approximate towards it the faſter, the greater b is, or the leſs d is; and that, the diameters being very ſmall, the bodies deſcend by nearly uniform velocities, which are directly in the ſubduplicate ratio of the diameters. See alſo art. 5, prob. 20, for other obſervations on this head.

5. To find the time t. Now $\dot{t} = \frac{\dot{s}}{v} = \sqrt{\frac{1}{a}} \times \frac{\dot{s}}{\sqrt{1 - c^{-2bs}}}$.

Then, to find the fluent of this fluxion, put $z = \sqrt{1 - c^{-2bs}} = \frac{v}{\sqrt{a}}$, or $z^2 = 1 - c^{-2bs}$; hence $z\dot{z} = b\dot{s}c^{-2bs}$, and $\dot{s} = \frac{z\dot{z}}{bc^{-2bs}} = \frac{1}{b} \cdot \frac{z\dot{z}}{1 - z^2}$; conſequently $\dot{t} = \frac{1}{b\sqrt{a}} \cdot \frac{\dot{z}}{1 - z^2}$, and therefore the fluent is $t = \frac{1}{2b\sqrt{a}} \times \log. \frac{1+z}{1-z} = \frac{1}{2b\sqrt{a}} \times \log. \frac{1 + \sqrt{1 - c^{-2bs}}}{1 - \sqrt{1 - c^{-2bs}}} = \frac{1}{2b\sqrt{a}} \times \log. \frac{\sqrt{a} + v}{\sqrt{a} - v}$, which is the general expreſſion for the time.

EXAM. If it were required to determine the time and velocity, by deſcending in air 1000 feet, the ball being of lead, and 1 inch diameter.

Here $N = 11\frac{1}{3}$, $n = \frac{3}{2500}$, $d = \frac{1}{12}$, and $s = 1000$.

Hence $a = \frac{2 \cdot 16\frac{1}{12} \cdot \frac{8}{10} \cdot 11\frac{1}{3}}{3 \cdot \frac{3}{2500}} = \frac{2 \cdot 193 \cdot 8 \cdot 34 \cdot 2500}{3 \cdot 3 \cdot 12 \cdot 12 \cdot 3} =$

$\frac{193\cdot34\cdot50^2}{9\cdot27}$, and $b = \frac{3\cdot\frac{3}{2500}}{8\cdot11\frac{1}{3}\cdot\frac{1}{12}} = \frac{3\cdot3\cdot3\cdot12}{8\cdot34\cdot2500} = \frac{9\cdot9}{68\cdot50^2}$;
consequently $v = \sqrt{a}\times\sqrt{1-c^{-2bs}} = \sqrt{\frac{193\cdot34\cdot50^2}{9\cdot27}}\times$
$\sqrt{1-c^{-\frac{81}{85}}} = 203\frac{2}{3}$ the velocity. And $t = \frac{1}{2b\sqrt{a}}\times$ log.
$\frac{1+\sqrt{1-c^{-2bs}}}{1-\sqrt{1-c^{-2bs}}} = \sqrt{\frac{34\cdot2500}{27\cdot193}}\times\log.\frac{1\cdot78383}{0\cdot21617} = 8\cdot5236''$,
the time.

Note. If the globe be so light as to ascend in the fluid; it is only necessary to change the signs of the first two terms in the value of f, or the accelerating force, by which it becomes $f = \frac{n}{N} - 1 - \frac{3nv^2}{16gNd}$; and then proceeding in all respects as before.

SCHOLIUM.

To compare this theory, contained in the last four problems, with experiment, the few following numbers are here extracted from extensive tables of velocities and resistances, resulting from a course of many hundred very accurate experiments, made in the course of the year 1786.

In the first column are contained the mean uniform or greatest velocities acquired in air, by globes, hemispheres, cylinders, and cones, all of the same diameter, and the altitude of the cone nearly equal to the diameter also, when urged by the several weights, expressed in averdupois ounces, and standing on the same line with the velocities, each in their proper column. So, in the first line, the numbers shew, that, when the greatest or uniform velocity was accurately 3 feet per second, the bodies were urged by these weights, according as their different ends went foremost; namely, by ·028 oz. when the vertex of the cone went foremost; by ·064 oz. when the base of the cone went foremost; by ·027 oz. for a whole sphere; by ·050 oz. for a cylinder; by ·051 oz. for the flat side of the hemisphere; and by ·020 oz. for the round or convex side of the hemisphere. Also, at the bottom of all, are placed the mean proportions of the resistances of these figures in the nearest whole numbers. Note, the common diameter of all the figures, was 6·375 or $6\frac{3}{8}$ inches; so that the area of the circle of that diameter is just 32 square inches, or $\frac{2}{9}$ of a square foot; and the altitude

tude of the cone was $6\frac{5}{8}$ inches. Alſo, the diameter of the ſmall hemiſphere was $4\frac{3}{4}$ inches, and conſequently the area of its baſe is $17\frac{3}{4}$ ſquare inches, or $\frac{1}{8}$ of a ſquare foot nearly.

From the given dimenſions of the cone, it appears, that the angle made by its ſide and axis, or direction of the path, is 26 degrees, very nearly.

The mean height of the barometer at the times of making the experiments, was nearly 30·1 inches, and of the thermometer 62°; conſequently the weight of a cubic foot of air was equal to $1\frac{1}{5}$ oz. nearly, in thoſe circumſtances.

Veloc. per ſec.	Cone.		Whole globe.	Cylinder.	Hemiſphere.		Small Hemiſ.
	vertex	baſe.			flat	round	flat
feet.	oz.	oz.	oz.	oz.	oz.	oz.	oz.
3	·028	·064	·027	·050	·051	·020	·028
4	·048	·109	·047	·090	·096	·039	·048
5	·071	·162	·068	·143	·148	·063	·072
6	·098	·225	·094	·205	·211	·092	·103
7	·129	·298	·125	·278	·284	·123	·141
8	·168	·382	·162	·360	·368	·160	·184
9	·211	·478	·205	·456	·464	·199	·233
10	·260	·587	·255	·565	·573	·242	·287
11	·315	·712	·310	·688	·698	·297	·349
12	·376	·850	·370	·826	·836	·347	·418
13	·440	1·000	·435	·979	·988	·409	·492
14	·512	1·166	·505	1·145	1·154	·478	·573
15	·589	1·346	·581	1·327	1·336	·552	·661
16	·673	1·546	·663	1·526	1·538	·634	·754
17	·762	1·763	·752	1·745	1·757	·722	·853
18	·858	2·002	·848	1·986	1·998	·818	·959
19	·959	2·260	·949	2·246	2·258	·922	1·073
20	1·069	2·540	1·05	2·528	2·542	1·033	1·196
Propor. Numb.	126	291	124	285	288	119	140

From this table of reſiſtances, ſeveral practical inferences may be drawn. As,

1. That the reſiſtance is nearly as the ſurface; the reſiſtance increaſing but a very little above that proportion in the greater

greater ſurfaces. Thus, by comparing together the numbers in the 6th and laſt columns, for the baſes of the two hemiſpheres, the areas of which are in the proportion of $17\frac{3}{4}$ to 32, or as 5 to 9 very nearly; it appears that the numbers in thoſe two columns, expreſſing the reſiſtances, are nearly as 1 to 2, or as 5 to 10, as far as to the velocity of 12 feet; after which the reſiſtances on the greater ſurface increaſe gradually more and more above that proportion. And the mean reſiſtances are as 140 to 288, or as 5 to $10\frac{2}{7}$. This circumſtance therefore agrees nearly with the theory.

2. The reſiſtance to the ſame ſurface, is nearly as the ſquare of the velocity; but gradually increaſes more and more above that proportion as the velocity increaſes. This is manifeſt from all the columns. And therefore this circumſtance alſo differs but little from the theory, in ſmall velocities.

3. When the hinder parts of bodies are of different forms, the reſiſtances are different, though the fore parts be alike; owing to the different preſſures of the air on the hinder parts. Thus, the reſiſtance to the fore part of the cylinder, is leſs than that on the flat baſe of the hemiſphere, or of the cone; becauſe the hinder part of the cylinder is more preſſed or puſhed, by the following air, than thoſe of the other two figures.

4. The reſiſtance on the baſe of the hemiſphere, is to that on the convex ſide, nearly as $2\frac{2}{5}$ to 1, inſtead of 2 to 1, as the theory aſſigns the proportion. And the experimented reſiſtance, in each of theſe, is nearly $\frac{1}{4}$ part more than that which is aſſigned by the theory.

5. The reſiſtance on the baſe of the cone is to that on the vertex, nearly as $2\frac{3}{10}$ to 1. And in the ſame ratio is radius to the ſine of the angle of the inclination of the ſide of the cone, to its path or axis. So that, in this inſtance, the reſiſtance is directly as the ſine of the angle of incidence, the tranſverſe ſection being the ſame, inſtead of the ſquare of the ſine.

6. Hence we can find the altitude of a column of air, whoſe preſſure ſhall be equal to the reſiſtance of a body, moving through it with any velocity. Thus,

Let a = the area of the ſection of the body, ſimilar to any of thoſe in the table, perpendicular to the direction of motion;

r = the reſiſtance to the velocity, in the table; and

x = the altitude ſought, of a column of air, whoſe baſe is a, and its preſſure r.

Then ax = the content of the column in feet, and $1\frac{1}{5}ax$ or $\frac{6}{5}ax$ its weight in ounces;

therefore $\frac{6}{5}ax = r$, and $x = \frac{5}{6} \times \frac{r}{a}$ is the altitude ſought in feet, namely, $\frac{5}{6}$ of the quotient of the reſiſtance of any body divided by its tranſverſe ſection; which is a conſtant quantity for all ſimilar bodies, however different in magnitude, ſince the reſiſtance r is as the ſection a, as was found in art. 1. When $a = \frac{2}{9}$ of a foot, as in all the figures in the foregoing table, except the ſmall hemiſphere: then, $x = \frac{5}{6} \times \frac{r}{a}$ becomes $x = \frac{15}{4}r$, where r is the reſiſtance in the table, to the ſimilar body.

If, for example, we take the convex ſide of the large hemiſphere, whoſe reſiſtance is ·634 oz. to a velocity of 16 feet per ſecond, then r = ·634, and $x = \frac{15}{4}r = 2{\cdot}3775$ feet, is the altitude of the column of air whoſe preſſure is equal to the reſiſtance on a ſpherical ſurface, with a velocity of 16 feet. And to compare the above altitude with that which is due to the given velocity, it will be $32^2 : 16^2 :: 16 : 4$, the altitude due to the velocity 16; which is near double the altitude that is equal to the preſſure. And as the altitude is proportional to the ſquare of the velocity, therefore, in ſmall velocities, the reſiſtance to any ſpherical ſurface, is equal to the preſſure of a column of air on its great circle, whoſe altitude is $\frac{19}{32}$ or ·594 of the altitude due to its velocity.

But if the cylinder be taken, whoſe reſiſtance $r = 1{\cdot}526$: then $x = \frac{15}{4}r = 5{\cdot}72$; which exceeds the height, 4, due to the velocity in the ratio of 23 to 16 nearly. And the difference would be ſtill greater, if the body were larger; and alſo if the velocity were more.

7. Alſo, if it be required to find with what velocity any flat ſurface muſt be moved, ſo as to ſuffer a reſiſtance juſt equal to the whole preſſure of the atmoſphere.

The

The resistance on the whole circle whose area is $\frac{2}{9}$ of a foot, is ·051 oz. with the velocity of 3 feet per second, it is $\frac{1}{9}$ of ·051, or ·0056 oz. only with a velocity of 1 foot. But $2\frac{1}{2} \times 13600 \times \frac{2}{9} = 7555\frac{5}{9}$ oz. is the whole pressure of the atmosphere. Therefore, as $\sqrt{\cdot 0056} : \sqrt{7556} :: 1 : 1162$ nearly, which is the velocity sought. Being almost equal to the velocity with which air rushes into a vacuum.

8. Hence may be inferred the great resistance suffered by military projectiles. For, in the table, it appears, that a globe of $6\frac{2}{8}$ inches diameter, which is equal to the size of an iron ball weighing 36lb, moving with a velocity of only 16 feet per second, meets with a resistance equal to the pressure of $\frac{2}{3}$ of an ounce weight; and therefore, computing only according to the square of the velocity, the least resistance that such a ball would meet with, when moving with a velocity of 1600 feet, would be equal to the pressure of 417lb, and that independent of the pressure of the atmosphere itself on the fore part of the ball, which would be 487lb more, as there would be no pressure from the atmosphere on the hinder part, in the case of so great a velocity as 1600 feet per second. So that the whole resistance would be more than 900lb to such a velocity.

9. Having said, in the last article, that the pressure of the atmosphere is taken entirely off the hinder part of the ball, moving with a velocity of 1600 feet per second; which must happen when the ball moves faster than the particles of air can follow by rushing into the place quitted and left void by the ball, or when the ball moves faster than the air rushes into a vacuum from the pressure of the incumbent air: let us therefore inquire what this velocity is. Now, the velocity with which any fluid issues, depends upon its altitude above the orifice, and is indeed equal to the velocity acquired by a heavy body in falling freely through that altitude. But, supposing the height of the barometer to be 30 inches, or $2\frac{1}{2}$ feet, the height of an uniform atmosphere, all of the same density as at the earth's surface, would be $2\frac{1}{2} \times 13{\cdot}6 \times 833\frac{1}{3}$ or 28333 feet; therefore $\sqrt{16} : \sqrt{28333} :: 32 : 8\sqrt{28333} = 1346$ feet, which is the velocity sought. And therefore, with a velocity of 1600 feet per second, or any velocity above 1346 feet, the ball must continually leave a vacuum behind it, and so must sustain the whole pressure of the atmosphere on its fore part, as well as the resistance arising from the vis inertia of the particles of air struck by the ball.

10. Upon

10. Upon the whole, we find, that the resistance of the air, as determined by the experiments, differs very widely, both in respect to the quantity of it on all figures, and in respect to the proportions of it on oblique surfaces, from the same as determined by the preceding theory; which is the same as that of Sir Isaac Newton, and most modern philosophers. Neither should we succeed better if we have recourse to the theory given by Professor Gravesande, or others, as similar differences and inconsistencies still occur.

We conclude, therefore, that all the theories of the resistance of the air hitherto given, are very erroneous. And the preceding one is only laid down, till further experiments, on this important subject, shall enable us to deduce from them, another, that shall be more consonant to the true phenomena of nature.

FINIS.

ERRATA.

Pa. 324, l. 14, for $\frac{ac}{r}$, read $\frac{cr}{a}$

Pa. 346, l. 9 from the bottom, for 1·36, read 1·96.

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